Canonical Automata via Distributive Law Homomorphisms

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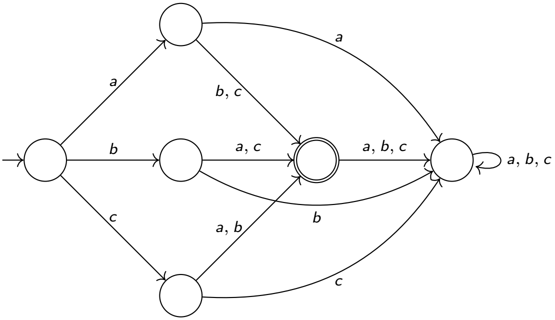
[1]Paper available at https://arxiv.org/abs/2104.13421

[2]Slides available at https://fgh.xyz

Introduction

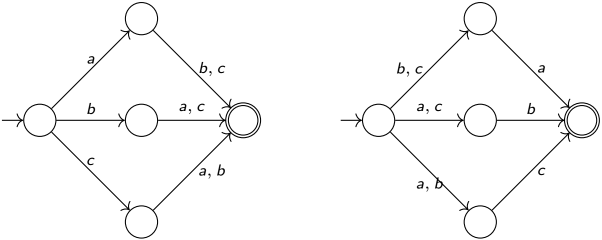
Minimal DFA

Up to isomorphism, the unique size-minimal DFA accepting $\mathcal{L} = \{ab, ac, ba, bc, ca, cb\} \subseteq \{a, b, c\}^*$:



Minimal NFA

Two non-isomorphic[3] size-minimal NFA accepting
$\mathcal{L} = \{ab, ac, ba, bc, ca, cb\} \subseteq \{a, b, c\}^*$:

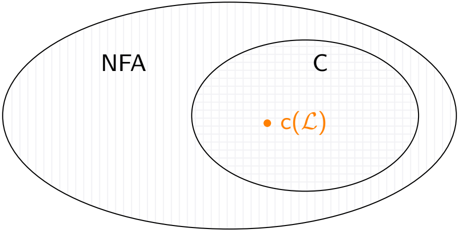


Is there a canonical NFA for $\mathcal{L}$?

[3] Arnold, Dicky, and Nivat 1992.

Minimal NFA

Is there a subclass $C \subseteq$ NFA, such that any regular language $\mathcal{L}$ admits a canonical acceptor $c(\mathcal{L}) \in C$ size-minimal in C?



Example: The canonical RFSA

A NFA accepting $\mathcal{L} \subseteq A^*$ is RFSA, if every state accepts a residual $u^{-1}\mathcal{L} = \{v \in A^* \mid uv \in \mathcal{L}\}$ for some $u \in A^*$.

The canonical RFSA for a regular language $\mathcal{L} \subseteq A^*$ is the X_0-pointed NFA $\langle \varepsilon, \delta \rangle : X \to 2 \times \mathcal{P}(X)^A$ given by:

- $X = \{U \subseteq A^* \mid U \text{ prime residual of } \mathcal{L}\}$;
- $X_0 = \{U \in X \mid U \subseteq \mathcal{L}\}$;
- $\varepsilon(U) = [\varepsilon \in U]$;
- $\delta_a(U) = \{V \in X \mid V \subseteq a^{-1}U\}$.

Theorem ([4])

The canonical RFSA for $\mathcal{L}$ is size-minimal among RFSA for $\mathcal{L}$.

[4] Denis, Lemay, and Terlutte 2002.

Example: The canonical RFSA

How does one come up with this definition? Why does it work?

NFA $\to$ DFA

The classical powerset construction converts a NFA into an equivalent DFA.

$$\langle \varepsilon, \delta \rangle : Y \to 2 \times \mathcal{P}(Y)^A$$

$$\downarrow$$

$$\langle \varepsilon^\sharp, \delta^\sharp \rangle : \mathcal{P}(Y) \to 2 \times \mathcal{P}(Y)^A$$

[5] $\varepsilon^\sharp(U) = \bigvee_{u \in U} \varepsilon(u), \quad \delta^\sharp_a(U) = \bigcup_{u \in U} \delta_a(u)$

NFA $\to$ DFA (in CSL)

$$\varepsilon^\sharp(U_1 \cup U_2) = \varepsilon^\sharp(U_1) \vee \varepsilon^\sharp(U_2)$$
$$\delta_a^\sharp(U_1 \cup U_2) = \delta_a^\sharp(U_1) \cup \delta_a^\sharp(U_2)$$

$\langle \varepsilon^\sharp, \delta^\sharp \rangle$ is a DFA in the category of complete semilattices (CSL).

DFA (in CSL) $\to$ NFA

Consider the reverse to the powerset construction.

$$\begin{array}{c} \langle E, D \rangle : X \to 2 \times X^A \\ \downarrow \\ \langle \varepsilon, \delta \rangle : Y \to 2 \times \mathcal{P}(Y)^A \end{array} \qquad 2, X \in \text{CSL}$$

Possible? Maybe, choose Y as a generator for X? Can we find a size-minimal generator Y?

[6] Constraint: $\langle E, D \rangle \sim \langle \varepsilon^\sharp, \delta^\sharp \rangle$

S-DFA $\to$ T-NFA

Generalises to other algebraic theories S, T:

$$\begin{array}{c} X \to B \times X^A \\ \downarrow \\ Y \to B \times T(Y)^A \end{array} \qquad B, X \in \mathsf{Alg}(S)$$

Related to the construction of canonical (minimal) automata:

name	S	T	B
canonical RFSA[7]	CSL	CSL	2
canonical nominal RFSA[8]	Nom-CSL	Nom-CSL	2
minimal xor automaton[9]	$\mathbb{Z}_2$-VSP	$\mathbb{Z}_2$-VSP	2
átomaton[10]	CABA	CSL	2
distromaton[11]	CDL	CSL	2

[7] Denis, Lemay, and Terlutte 2002.

[8] Moerman and Sammartino 2020.

[9] Vuillemin and Gama 2010.

[10] Brzozowski and Tamm 2014.

[11] Myers et al. 2015.

Plan

Preliminaries

- Distributive laws, bialgebras

Diagonal cases ($S = T$)

- Generators for (bi)algebras
- Example: The canonical RFSA

Extension to non-diagonal cases ($S \neq T$)

- (Deriving) distributive law homomorphisms
- Example: The átomaton
- The minimal xor-CABA automaton
- Minimality results

Preliminaries

Overview

We make the following generalisations:

CSL	$TX \to X \in \mathsf{Alg}(T)$
DFA	$X \to FX \in \mathsf{Coalg}(F)$
S-DFA	$SX \to X \to FX \in \mathsf{Bialg}(\lambda^S)$
T-NFA	$T^2Y \to TY \to FTY \in \mathsf{Bialg}(\lambda^T)$

Distributive laws

A distributive law between a monad $\langle T, \eta, \mu \rangle$ on C and an endofunctor $F : \mathsf{C} \to \mathsf{C}$ is a natural transformation

$$\lambda : TF \Rightarrow FT$$

satisfying the laws:

$$\begin{array}{ccc} FX & \xrightarrow{F\eta_X} & FTX \\ {\scriptstyle \eta_{FX}}\downarrow & \nearrow{\scriptstyle \lambda_X} & \\ TFX & & \end{array} \qquad \begin{array}{ccccc} T^2FX & \xrightarrow{T\lambda_X} & TFTX & \xrightarrow{\lambda_{TX}} & FT^2X \\ {\scriptstyle \mu_{FX}}\downarrow & & & & \downarrow{\scriptstyle F\mu_X} \\ TFX & & \xrightarrow{\quad\lambda_X\quad} & & FTX \end{array} .$$

[12]For example, if F satisfies $FX = B \times X^A$ and $\langle B, h \rangle$ is a T-algebra, the family

$$(\lambda^h)_X : TFX = T(B \times X^A) \overset{\langle T\pi_1, T\pi_2 \rangle}{\to} TB \times T(X^A) \overset{h \times \mathrm{st}}{\to} B \times (TX)^A = FTX$$

induces a distributive law between T and F.

Bialgebras

A λ-bialgebra is a tuple $\langle X, h, k \rangle$ consisting of an object X and morphisms

$$TX \xrightarrow{h} X \in \mathrm{Alg}(T), \qquad X \xrightarrow{k} FX \in \mathrm{Coalg}(F)$$

satisfying:

$$\begin{array}{ccccc} TX & & \xrightarrow{\quad h \quad} & & X \\ \downarrow{\scriptstyle Tk} & & & & \downarrow{\scriptstyle k} \\ TFX & \xrightarrow{\lambda_X} & FTX & \xrightarrow{Fh} & FX \end{array} \; .$$

Diagonal cases

Generators

A generator for a T-algebra $\langle X, h \rangle$ is a tuple $\langle Y, i, d \rangle$ consisting of an object Y and a pair of morphisms

$$TY \underset{d}{\overset{i^\sharp}{\rightleftarrows}} X \qquad \text{satisfying} \qquad i^\sharp \circ d = \mathrm{id}_X,$$

where $i^\sharp := h \circ Ti : TY \to X$ is the unique extension of $i : Y \to X$ to a T-algebra homomorphism.

If in addition $d \circ i^\sharp = \mathrm{id}_{TY}$, we speak of a basis.

Generators

$\langle Y, i, d \rangle$ is a generator for an algebra $\langle X, h \rangle$ over the powerset monad iff for all $x \in X$:

$$x = \bigvee_{y \in Y}^{h} d(x)(y) \cdot^{h} i(y).$$

$\langle Y, i, d \rangle$ is a generator for an algebra $\langle X, h \rangle$ over the free $\mathbb{Z}_2$-vector-space monad iff for all $x \in X$:

$$x = \bigoplus_{y \in Y}^{h} d(x)(y) \cdot^{h} i(y).$$

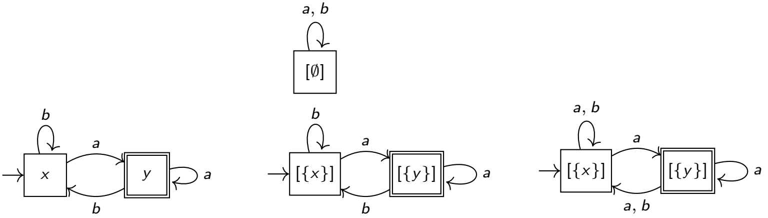
Generators

Let $\langle X, h, k \rangle$ be a λ-bialgebra and $\langle Y, i, d \rangle$ a generator for the T-algebra $\langle X, h \rangle$.

Lemma
The morphism $i^\sharp = h \circ Ti : TY \to X$ is a λ-bialgebra homomorphism

$$i^\sharp : \langle TY, \mu_Y, (Fd \circ k \circ i)^\sharp \rangle \to \langle X, h, k \rangle.$$

Example: The canonical RFSA



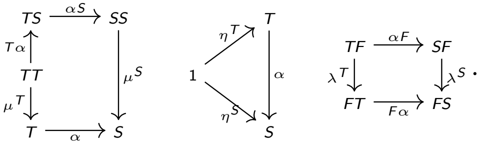
(a) (b) (c)

Figure:
(a) The minimal DFA for $\mathcal{L} = (a+b)^*a$;
(b) The minimal CSL-structured DFA $\langle X, h, k\rangle$ for $\mathcal{L}$;
(c) The canonical RFSA $\langle J(\langle X, h\rangle), Fd \circ k \circ i\rangle$ for $\mathcal{L}$.

Extension to non-diagonal cases

Distributive law homomorphisms

A distributive law homomorphism[13] $\alpha : \lambda^S \to \lambda^T$ between $\lambda^S : SF \Rightarrow FS$ and $\lambda^T : TF \Rightarrow FT$ consists of a natural transformation $\alpha : T \Rightarrow S$ satisfying:



Lemma ([14])

Defining $\alpha\langle X, h, k\rangle := \langle X, h \circ \alpha_X, k\rangle$ *and* $\alpha(f) := f$ *yields a functor* $\alpha : \mathrm{Bialg}(\lambda^S) \to \mathrm{Bialg}(\lambda^T)$.

[13] Watanabe 2002; Power and Watanabe 2002.

[14] Klin and Nachyla 2015; Bonsangue et al. 2013.

Distributive law homomorphisms

The following can be seen as roadmap or soundness argument to our approach.

Corollary

Let $\alpha : \lambda^S \to \lambda^T$ be a homomorphism between distributive laws and $\langle X, h, k\rangle$ a λ^S-bialgebra. If $\langle Y, i, d\rangle$ is a generator for the T-algebra $\langle X, h \circ \alpha_X\rangle$, then:

$$(h \circ \alpha_X) \circ Ti : \langle TY, \mu_Y, (Fd \circ k \circ i)^\sharp\rangle \to \langle X, h \circ \alpha_X, k\rangle$$

is a λ^T-bialgebra homomorphism.

[15] In consequence, $Fd \circ k \circ i : Y \to FTY$ is semantically equivalent to $k : X \to FX$.

Deriving distributive law homomorphisms

If the distributive laws are induced by algebras $h^S : SB \to B$ and $h^T : TB \to B$, respectively, then deriving a homomorphism simplifies.

Lemma

Let $\alpha : T \to S$ be a natural transformation satisfying $h^S \circ \alpha_B = h^T$, then:

$$\begin{array}{ccc} TF & \xrightarrow{\alpha F} & SF \\ \lambda^{h^T}\downarrow & & \downarrow\lambda^{h^S} \\ FT & \xrightarrow[F\alpha]{} & FS \end{array} .$$

[16] $(\lambda^h)_X : TFX = T(B \times X^A) \overset{\langle T\pi_1, T\pi_2\rangle}{\to} TB \times T(X^A) \overset{h\times \mathrm{st}}{\to} B \times (TX)^A = FTX$

Deriving distributive law homomorphisms

For the neighbourhood monad $\mathcal{H}$, there exists a parametrised family of canonical homomorphisms:

Corollary

Any algebra $h : T2 \to 2$ *over a set monad* T *induces a homomorphism* $\alpha^h : \lambda^{\mathcal{H}} \to \lambda^h$ *between distributive laws by* $\alpha^h_X := h^{2^X} \circ \mathsf{st} \circ T(\eta^{\mathcal{H}}_X) : TX \to \mathcal{H}X$.

Can be lifted to strong monads and arbitrary output objects on general categories.

[17] $\mathcal{H}X = 2^{2^X}$, $\mathcal{H}f(\Phi)(\varphi) = \Phi(\varphi \circ f)$, $\eta^{\mathcal{H}}_X(x)(\varphi) = \varphi(x)$, $\mu^{\mathcal{H}}_X(\Psi)(\varphi) = \Psi(\eta^{\mathcal{H}}_{2^X}(\varphi))$, $\mathsf{Alg}(\mathcal{H}) = \mathsf{CABA}$.

[18] $h^{\mathcal{H}} : \mathcal{H}2 \to 2, \Phi \mapsto \Phi(\mathsf{id}_2)$, $\lambda^{\mathcal{H}} := \lambda^{h^{\mathcal{H}}}$

Example: The átomaton

The átomaton can be recovered by relating $\mathcal{H}$ (CABA) and $\mathcal{P}$ (CSL).

Corollary

Let $\alpha_X : \mathcal{P}X \to \mathcal{H}X$ *satisfy* $\alpha_X(\varphi)(\psi) = \bigvee_{x \in X} \varphi(x) \wedge \psi(x)$*, then* α *constitutes a distributive law homomorphism* $\alpha : \lambda^{\mathcal{H}} \to \lambda^{\mathcal{P}}$*.*

Lemma

If $B = \langle X, h \rangle$ *is a* $\mathcal{H}$*-algebra, then* $\langle \mathrm{At}(B), i, d \rangle$ *with* $i(a) = a$ *and* $d(x) = \{a \in \mathrm{At}(B) \mid a \leq x\}$ *is a basis for the* $\mathcal{P}$*-algebra* $\langle X, h \circ \alpha_X \rangle$*.*

Example: The átomaton

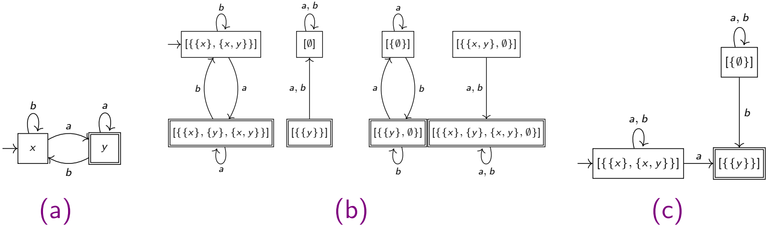
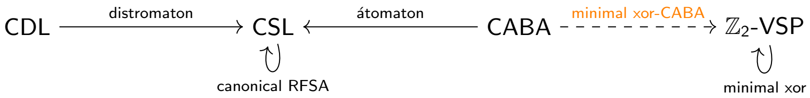


Figure:
(a) The minimal DFA for $\mathcal{L} = (a + b)^* a$;
(b) The minimal CABA-structured DFA $\langle X, h, k \rangle$ for $\mathcal{L}$;
(c) The átomaton $\langle \mathrm{At}(\langle X, h \rangle), Fd \circ k \circ i \rangle$ for $\mathcal{L}$.

The minimal xor-CABA automaton

"The minimal xor-CABA automaton is to the minimal xor automaton what the átomaton is to the canonical RFSA":



Definition

Let $\alpha : \lambda^S \to \lambda^T$ be a distributive law homomorphism. We say $\mathcal{X} \in \mathrm{Coalg}(FT)$ is *α-closed* if the unique diagonal below is an isomorphism:

$$\begin{array}{ccc}
\exp_T(\mathcal{X}) & \xrightarrow{\;\mathrm{obs}\;} & \mathrm{im}(\mathrm{obs}_{\exp_T(\mathcal{X})}) \\
\Big\downarrow{\scriptstyle \mathrm{obs}\circ\alpha_X} & \swarrow & \Big\downarrow \\
\mathrm{im}(\mathrm{obs}_{\alpha(\exp_S(\mathrm{ext}(\mathcal{X})))}) & \hookrightarrow & \Omega
\end{array} .$$

[19] $\mathrm{ext} : \mathrm{Coalg}(FT) \to \mathrm{Coalg}(FS)$

[20] $\exp_U : \mathrm{Coalg}(FU) \to \mathrm{Bialg}(\lambda^U)$ for $U \in \{S, T\}$

[21] $\alpha : \mathrm{Bialg}(\lambda^S) \to \mathrm{Bialg}(\lambda^T)$

Minimality

Theorem
Given a minimal λ^S-bialgebra $\mathbb{M}$ accepting $\mathcal{L}$ such that $\alpha(\mathbb{M})$ admits a size-minimal generator, there exists $\mathcal{X} \in \mathsf{Coalg}(FT)$ accepting $\mathcal{L}$ such that:

- *for any α-closed $\mathcal{Y} \in \mathsf{Coalg}(FT)$ accepting $\mathcal{L}$ we have that $\mathsf{im}(\mathsf{obs}_{\mathsf{exp}_T(\mathcal{X})}) \subseteq \mathsf{im}(\mathsf{obs}_{\mathsf{exp}_T(\mathcal{Y})})$;*
- *if $\mathsf{im}(\mathsf{obs}_{\mathsf{exp}_T(\mathcal{X})}) = \mathsf{im}(\mathsf{obs}_{\mathsf{exp}_T(\mathcal{Y})})$, then $|\mathcal{X}| \leq |\mathcal{Y}|$.*

Minimality

Lemma

$\mathcal{X}$ is minimal among $\mathcal{Y}$ that accept $\mathcal{L}$ and satisfy:

$\mathcal{X}$	$\mathcal{Y}$
canonical RFSA	$\mathsf{CSL}[\mathcal{Y}] = \mathsf{CSL}[\mathsf{Der}(\mathcal{L})]$
minimal xor automaton	all
átomaton	$\mathsf{CSL}[\mathcal{Y}] = \mathsf{CABA}[\mathcal{Y}]$
distromaton	$\mathsf{CSL}[\mathcal{Y}] = \mathsf{CDL}[\mathcal{Y}]$
minimal xor-CABA automaton	$\mathbb{Z}_2\text{-}\mathsf{Vect}[\mathcal{Y}] = \mathsf{CABA}[\mathcal{Y}]$

[22] $T[\mathcal{Y}]$ denotes the T-closure of $\mathrm{im}(\mathrm{obs}_{\exp_T(\mathcal{Y})})$.

Future work

Ideas

- Cover the canonical probabilistic RFSA[23] and canonical alternating RFSA[24];
- Utilise distributive laws between two different categories;
- Generalise Brzozowski[25] inspired double reversal characterisations;
- Further explore the notions of generators and bases.

[23] Esposito et al. 2002.

[24] Berndt et al. 2017.

[25] Brzozowski 1962.

The end

Thanks for listening!

[25] Accepted at MFPS 2021

[26] Paper available at https://arxiv.org/abs/2104.13421

[27] Slides available at https://fgh.xyz