Bases for algebras over a monad

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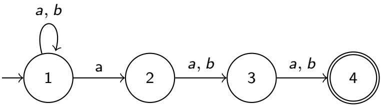
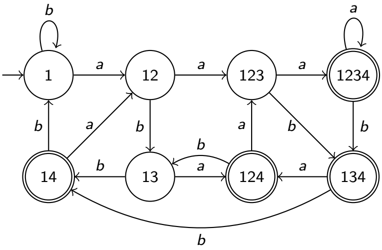
[0]https://arxiv.org/abs/2010.10223

Motivation: NFA, DFA

A NFA accepting L^1:



Up to iso, the minimal DFA accepting L:



[1] $L = \{w \in \{a, b\}^* \mid |w| \geq 3$ and the 3^{rd} symbol from the right is a$\}$

Motivation: NFA $\to$ DFA

$$\langle \delta, \varepsilon \rangle : Y \to \mathcal{P}(Y)^A \times 2$$

$$\downarrow$$

$$\langle \delta^\sharp, \varepsilon^\sharp \rangle^2 : \mathcal{P}(Y) \to \mathcal{P}(Y)^A \times 2$$

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[2] $\delta^\sharp_a(U) = \bigcup_{u \in U} \delta_a(u), \quad \varepsilon^\sharp(U) = \bigvee_{u \in U} \varepsilon(u)$

Motivation: NFA $\rightarrow$ DFA (in CSL)

$$\delta_a^\sharp(U_1 \cup U_2) = \delta_a^\sharp(U_1) \cup \delta_a^\sharp(U_2)$$
$$\varepsilon^\sharp(U_1 \cup U_2) = \varepsilon^\sharp(U_1) \vee \varepsilon^\sharp(U_2)$$

$\langle \delta, \varepsilon \rangle$ is a NFA in the category of sets
$\langle \delta^\sharp, \varepsilon^\sharp \rangle$ is a DFA in the category of complete semilattices

[2] $\delta_a^\sharp(U) = \bigcup_{u \in U} \delta_a(u), \quad \varepsilon^\sharp(U) = \bigvee_{u \in U} \varepsilon(u)$

Motivation: DFA (in CSL) $\to$ NFA

$$\langle D, E\rangle : L \to L^A \times 2$$

$$\Big\downarrow {\scriptstyle 3}$$

$$\langle \delta, \varepsilon\rangle : Y \to \mathcal{P}(Y)^A \times 2$$

Possible? Maybe, choose Y as a generator for L? Can we find a minimal Y?

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[3] Constraint: $\langle D, E\rangle \sim \langle \delta^\sharp, \varepsilon^\sharp\rangle$

Motivation: DFA (in CSL) $\to$ NFA

Let L be a join semi-lattice.

A subset $Y \subseteq L$ is join-dense in L iff for all $x \in L$ there exists a decomposition

$$x = y_1 \vee ... \vee y_n,$$

where $y_i \in Y$ for $i = 1, ..., n$.

If L is finite or satisfies the descending chain condition, the set of join-irreducibles $J(L)$[4] is join-dense in L.

[4] $x \in J(L)$ iff $x \neq 0$ and $\forall y, z \in L$: $x = y \vee z$ implies $x = y$ or $x = z$.

Motivation: DFA (in ?) $\to$?

$$\begin{array}{ccc} L \to L^A \times 2 & \qquad & V \to V^A \times 2 \\ \downarrow & & \downarrow \\ Y \to T_{\mathtt{CSL}}(Y)^A \times 2 & & Y \to T_{\mathtt{VSP}}(Y)^A \times 2 \end{array}$$

[4] $T_{\mathtt{CSL}} = \mathcal{P}$, $T_{\mathtt{VSP}} = ?$

Preliminaries

Algebra in theory T	$TX \to X \in \mathrm{Alg}(T)$
Free algebra in theory T	$T^2 Y \to TY \in \mathrm{Alg}(T)$
DFA in Set	$X \to FX \in \mathrm{Coalg}(F)$
DFA in CSL	$TX \to X \to FX \in \mathrm{Bialg}(\lambda)$
NFA in Set	$T^2 Y \to TY \to FTY \in \mathrm{Bialg}(\lambda)$

Preliminaries: Monads

A monad is a tuple $\langle T, \eta, \mu \rangle$ consisting of an endofunctor $T : \mathsf{C} \to \mathsf{C}$ and natural transformations

$$\eta : 1 \Rightarrow T \qquad \mu : T^2 \Rightarrow T$$

satisfying

$$\mu \circ \eta_T = 1 = \mu \circ T\eta \qquad \mu \circ T\mu = \mu \circ \mu_T.$$

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[4]For instance, the powerset monad with

$$T_{\mathrm{CSL}} X = 2^X, \quad \eta_X(x)(y) = [x = y], \quad \mu_X(\Phi)(x) = \bigvee_{\varphi \in 2^X} \Phi(\varphi) \wedge \varphi(x);$$

and the free vector space monad with

$$T_{\mathrm{VSP}} X = k^X|_{\mathsf{fs}}, \quad \eta_X(x)(y) = [x = y], \quad \mu_X(\Phi)(x) = \sum_{\varphi \in k^X} \Phi(\varphi) \cdot \varphi(x).$$

Preliminaries: Algebras over a monad

An algebra over a monad $\langle T, \eta, \mu \rangle$ is a tuple $\langle X, h \rangle$ consisting of a morphism

$$h : TX \to X$$

satisfying

$$h \circ \eta_X = \mathrm{id}_X \qquad h \circ Th = h \circ \mu_X .$$

[4] For instance, there are equivalences

$$\mathrm{Alg}(T_{\mathrm{CSL}}) \simeq \mathrm{CSL} \qquad \mathrm{Alg}(T_{\mathrm{VSP}}) \simeq \mathrm{VSP}.$$

Preliminaries: Distributive laws

A distributive law between a monad $\langle T, \eta, \mu \rangle$ and an endofunctor F is a natural transformation

$$\lambda : TF \Rightarrow FT$$

satisfying the laws

$$\lambda \circ \eta_F = F\eta \qquad \lambda\mu_F = F\mu \circ \lambda_T \circ T\lambda.$$

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[4] For example, if F satisfies $FX = X^A \times B$ and $\langle B, h \rangle$ is a T-algebra,

$$\lambda_X : T(X^A \times B) \overset{\langle T\pi_1, T\pi_2 \rangle}{\to} T(X^A) \times TB \overset{\mathrm{st} \times h}{\to} (TX)^A \times B$$

gives rise to a distributive law between T and F.

Preliminaries: Distributive laws

There exist liftings T_λ and F_λ

$$
\begin{array}{ccc}
\mathrm{Coalg}(F) & \xrightarrow{T_\lambda} & \mathrm{Coalg}(F) \\
{\scriptstyle U_F}\big\downarrow & & \big\downarrow{\scriptstyle U_F} \\
\mathsf{C} & \xrightarrow{\quad T \quad} & \mathsf{C}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathrm{Alg}(T) & \xrightarrow{F_\lambda} & \mathrm{Alg}(T) \\
{\scriptstyle U_T}\big\downarrow & & \big\downarrow{\scriptstyle U_T} \\
\mathsf{C} & \xrightarrow{\quad F \quad} & \mathsf{C}
\end{array}
$$

satisfying

$$T_\lambda(X \stackrel{k}{\to} FX) = TX \stackrel{\lambda_X \circ Tk}{\to} FTX$$
$$F_\lambda(TX \stackrel{h}{\to} X) = TFX \stackrel{Fh \circ \lambda_X}{\to} FX.$$

[4] In fact, liftings of T to $\mathrm{Coalg}(F)$, liftings of F to $\mathrm{Alg}(T)$, and distributive laws coincide.

Preliminaries: Bialgebras

A λ-bialgebra is an object with both a T-algebra and a F-coalgebra structure

$$\langle TX \xrightarrow{h} X \xrightarrow{k} FX \rangle,$$

satisfying

$$\begin{array}{ccccc} TX & & \xrightarrow{h} & & X \\ \downarrow{\scriptstyle Tk} & & & & \downarrow{\scriptstyle k} \\ TFX & \xrightarrow{\lambda_X} & FTX & \xrightarrow{Fh} & FX \end{array} .$$

There exist equivalences

$$\mathrm{Alg}(T_\lambda) \simeq \mathrm{Bialg}(\lambda) \simeq \mathrm{Coalg}(F_\lambda).$$

Overview

- Generators for algebras
- Bases for algebras
- Bases for bialgebras
- Basis representation
- Alternative approach
- Future work

Generators

A generator[5] for a T-algebra $\langle X, h \rangle$ is a tuple $\langle Y, i, d \rangle$ consisting of an object Y and morphisms

$$i : Y \to X \qquad d : X \to TY$$

satisfying

$$\begin{array}{ccc} TY & \xrightarrow{Ti} & TX \\ {\scriptstyle d}\uparrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{\mathrm{id}_X} & X \end{array} .$$

[5] Arbib and Manes, "Fuzzy machines in a category".
[5] For instance, every T-algebra $\langle X, h \rangle$ is generated by $\langle X, \mathrm{id}_X, \eta_X \rangle$.

Generators

$\langle Y, i, d \rangle$ is a generator for a T_{CSL}-algebra $\langle X, h \rangle$ iff for all $x \in X$

$$x = \bigvee_{y \in d(x)}^{h} i(y).$$

$\langle Y, i, d \rangle$ is a generator for a T_{VSP}-algebra $\langle X, h \rangle$ iff for all $x \in X$

$$x = \sum_{y \in Y}^{h} d(x)(y) \cdot^{h} i(y).$$

[5] $i : Y \to X$, $d : X \to TY$

Generators

Let $\langle X, h, k \rangle$ be a λ-bialgebra and $\langle Y, i, d \rangle$ a generator for the T-algebra $\langle X, h \rangle$.

Lemma

The morphism $h \circ Ti : TY \to X$ is a λ-bialgebra homomorphism

$$h \circ Ti : \langle TY, \mu_Y, (Fd \circ k \circ i)^{\sharp} \rangle \to \langle X, h, k \rangle.$$ [6]

[6] $(Fd \circ k \circ i)^{\sharp} := F\mu_Y \circ \lambda_{TY} \circ T(Fd \circ k \circ i)$

Generators

Let λ be the canonical[7] distributive law between T_{CSL} and F with $FX = X^A \times 2$.

Let $\langle X, h, k \rangle$ be the minimal λ-bialgebra accepting a regular language L.

Then $\langle J(X), i, d \rangle$ with $i(y) = y$ and $d(x) = \{y \in J(X) \mid y \leq x\}$ is a generator for $\langle X, h \rangle$.

The induced non-deterministic automaton

$$J(X) \xrightarrow{i} X \xrightarrow{k} FX \xrightarrow{Fd} FT_{\mathrm{CSL}}(J(X))$$

is given by the so-called canonical residual finite state automaton[8] for L.

[7] Induced by the T_{CSL}-algebra $\langle 2, h \rangle$ with $h(U) = \bigvee_{u \in U} u$.

[8] Denis, Lemay, and Terlutte, "Residual finite state automata".

Bases

A basis for a T-algebra $\langle X, h \rangle$ is a tuple $\langle Y, i, d \rangle$ consisting of an object Y, a morphism $i : Y \to X$, and a morphism $d : X \to TY$, satisfying

$$\begin{array}{ccc} TY & \xrightarrow{Ti} & TX \\ {\scriptstyle d}\uparrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{\mathrm{id}_X} & X \end{array} \qquad \begin{array}{ccc} TX & \xrightarrow{h} & X \\ {\scriptstyle Ti}\uparrow & & \downarrow{\scriptstyle d} \\ TY & \xrightarrow{\mathrm{id}_{TY}} & TY \end{array} .$$

[8]A free T-algebra $\langle TX, \mu_X \rangle$ has the basis $\langle X, \eta_X, \mathrm{id}_{TX} \rangle$. In fact, a T-algebra admits a basis iff it is isomorphic to a free T-algebra.

Bases

Let $\langle Y, i, d\rangle$ be a basis for a T-algebra $\langle X, h\rangle$.

Lemma

The following two diagrams commute

$$
\begin{array}{ccc}
TX & \xrightarrow{Td} & T^2Y \\
{\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle \mu_Y} \\
X & \xrightarrow{d} & TY
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y & \xrightarrow{i} & X \\
{\scriptstyle \eta_Y}\big\downarrow & \swarrow{\scriptstyle d} & \\
TY & &
\end{array}
\quad .
$$

Corollary

A T-algebra homomorphism is uniquely determined by its restriction to a basis

$$
\begin{array}{ccc}
\langle X, h\rangle & \overset{f^\sharp}{\dashrightarrow} & \langle Z, h_Z\rangle \\
{\scriptstyle i}\big\uparrow & \nearrow{\scriptstyle f} & \\
Y & &
\end{array}
\quad .
$$

Bases

Let $\langle X, h, k \rangle$ be a λ-bialgebra and $\langle Y, i, d \rangle$ be a basis for the T-algebra $\langle X, h \rangle$.

Lemma

The morphism $d : X \to TY$ is a λ-bialgebra homomorphism

$$d : \langle X, h, k \rangle \to \langle TY, \mu_Y, (Fd \circ k \circ i)^\sharp \rangle.$$

Corollary

The morphism $h \circ Ti : TY \to X$ is a λ-bialgebra isomorphism

$$h \circ Ti : \langle TY, \mu_Y, (Fd \circ k \circ i)^\sharp \rangle \to \langle X, h, k \rangle.$$

Bases for bialgebras

Recall the equivalence

$$\mathsf{Bialg}(\lambda) \simeq \mathsf{Alg}(T_\lambda : \mathsf{Coalg}(F) \to \mathsf{Coalg}(F)).$$

Lemma

Let $\langle Y, k_Y, i, d \rangle$ be a generator for a T_λ-algebra $\langle X, h, k \rangle$, then the morphism $h \circ Ti : TY \to X$ is a λ-bialgebra homomorphism

$$h \circ Ti : \langle TY, \mu_Y, \lambda_Y \circ Tk_Y \rangle \to \langle X, h, k \rangle.$$

Lemma

Let $\langle Y, k_Y, i, d \rangle$ be a basis for a T_λ-algebra $\langle X, h, k \rangle$, then

$$\lambda_Y \circ Tk_Y = (Fd \circ k \circ i)^\sharp.$$

Basis representation

Assume the following data

$$\alpha = \{\alpha_1, ..., \alpha_n\} \text{ : basis for the } k\text{-vector space } V$$
$$\beta = \{\beta_1, ..., \beta_m\} \text{ : basis for the } k\text{-vector space } W.$$

Every linear transformation $L : V \to W$ admits a representation $L_{\alpha\beta} \in \mathrm{Mat}_k(m, n)$ with

$$L(\alpha_j) = \sum_i (L_{\alpha\beta})_{i,j} \cdot \beta_i,$$

such that the coordinate vectors[9] satisfy the matrix product equality

$$L(v)_\beta = L_{\alpha\beta} v_\alpha.$$

[9] $v = \sum_i (v_\alpha)_i \cdot \alpha_i$

Basis representation

Assume the following data

$$\alpha = \langle Y_\alpha, i_\alpha, d_\alpha \rangle \text{ : basis for the } T\text{-algebra } \langle X_\alpha, h_\alpha \rangle$$
$$\beta = \langle Y_\beta, i_\beta, d_\beta \rangle \text{ : basis for the } T\text{-algebra } \langle X_\beta, h_\beta \rangle.$$

Given a T-algebra homomorphism $f : \langle X_\alpha, h_\alpha \rangle \to \langle X_\beta, h_\beta \rangle$, we define

$$f_{\alpha\beta} := Y_\alpha \xrightarrow{i_\alpha} X_\alpha \xrightarrow{f} X_\beta \xrightarrow{d_\beta} TY_\beta. \tag{1}$$

Given a morphism $p : Y_\alpha \to TY_\beta$, we define

$$p^{\alpha\beta} := X_\alpha \xrightarrow{d_\alpha} TY_\alpha \xrightarrow{Tp} T^2 Y_\beta \xrightarrow{\mu_{Y_\beta}} TY_\beta \xrightarrow{Ti_\beta} TX_\beta \xrightarrow{h_\beta} X_\beta. \tag{2}$$

Basis representation

Lemma

The morphism $p^{\alpha\beta} : X_\alpha \to X_\beta$ *is a* T*-algebra homomorphism*

$$p^{\alpha\beta} : \langle X_\alpha, h_\alpha \rangle \to \langle X_\beta, h_\beta \rangle.$$

Lemma

The operations (1) *and* (2) *are mutually inverse,*

$$(f_{\alpha\beta})^{\alpha\beta} = f \qquad (p^{\alpha\beta})_{\alpha\beta} = p.$$

Lemma

The operations (1) *and* (2) *are compositional*[10]*,*

$$g_{\beta\gamma} \cdot f_{\alpha\beta} = (g \circ f)_{\alpha\gamma} \qquad q^{\beta\gamma} \circ p^{\alpha\beta} = (q \cdot p)^{\alpha\gamma}.$$

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[10] $q \cdot p := \mu_{Y_\gamma} \circ Tq \circ p$

Basis representation

Assume the following data

$$\begin{aligned} \alpha, \alpha' &: \text{bases for the } T\text{-algebra } \langle X_\alpha, h_\alpha \rangle \\ \beta, \beta' &: \text{bases for the } T\text{-algebra } \langle X_\beta, h_\beta \rangle \\ f &: \langle X_\alpha, h_\alpha \rangle \to \langle X_\beta, h_\beta \rangle. \end{aligned}$$

Lemma

There exist Kleisli isomorphisms p and q such that

$$f_{\alpha'\beta'} = q \cdot f_{\alpha\beta} \cdot p.$$

Basis representation

Assume the following data

$$\alpha, \alpha' : \text{bases for the } T\text{-algebra } \langle X_\alpha, h_\alpha \rangle$$
$$f : \langle X_\alpha, h_\alpha \rangle \to \langle X_\alpha, h_\alpha \rangle.$$

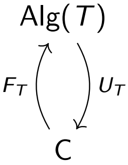
Corollary

There exists a Kleisli isomorphism p with Kleisli inverse p^{-1} such that

$$f_{\alpha'\alpha'} = p^{-1} \cdot f_{\alpha\alpha} \cdot p.$$

Alternative approach

Let $T : \mathsf{C} \to \mathsf{C}$ be a monad. The adjunction



incduces a comonad $\overline{T} = F_T \circ U_T : \mathsf{Alg}(T) \to \mathsf{Alg}(T)$.

A BASIS[11] for a T-algebra $\langle X, h \rangle$ is a $\overline{T}$-coalgebra

$$k : \langle X, h \rangle \to \overline{T}\langle X, h \rangle.$$

[11] Jacobs, "Bases as coalgebras".

Alternative approach

Let $\langle Y, i, d \rangle$ be a basis for a T-algebra $\langle X, h \rangle$.

Lemma

The morphism $Ti \circ d : X \to TX$ is a BASIS for $\langle X, h \rangle$.

Conversely, under certain assumptions[12], it is possible to recover a basis from a BASIS.

[12] If there exists an equaliser of k and η_X, that is preserved by T.

Future work 1

Let $\langle D, \varepsilon, \delta \rangle$ be a comonad.

A cogenerator for a D-coalgebra $\langle X, k \rangle$ is a tuple $\langle Y, i, d \rangle$ consisting of an object Y, a morphism $i : X \to Y$, and a morphism $d : DY \to X$, satisfying

$$\begin{array}{ccc} DY & \xleftarrow{Di} & DX \\ {\scriptstyle d}\downarrow & & \uparrow{\scriptstyle k} \\ X & \xleftarrow{\mathrm{id}_X} & X \end{array} .$$

[12] For example, let $\langle \hat{\delta}, \varepsilon \rangle : X \to X^{A^*} \times 2$ be a DFA such that the final coalgebra semantics $[\cdot] : X \to 2^{A^*}$ admits a left-inverse d. Then $\langle 2, \varepsilon, d \rangle$ is a cogenerator for the coalgebra $\langle X, \hat{\delta} \rangle$ of the comonad D with $DX = X^{A^*}$.

Future work 2

Let $T_{\texttt{CABA}}$ be the neighbourhood monad with $T_{\texttt{CABA}}X = 2^{2^X}$, then

$$\mathsf{Alg}(T_{\texttt{CABA}}) \simeq \mathsf{CABA}.$$

Moreover, for every complete atomic boolean algebra B,

$$B \simeq 2^{\mathsf{At}(B)}.$$

$\mathsf{At}(B)$ is not a $T_{\texttt{CABA}}$-basis for B, so what is it? Maybe, use a definition parametric in two monads?

The end

Thanks!