# Canonical automata via distributive law homomorphisms

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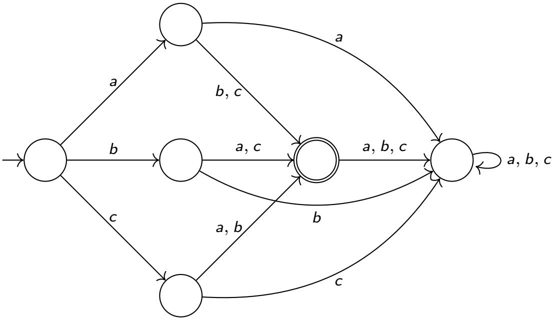
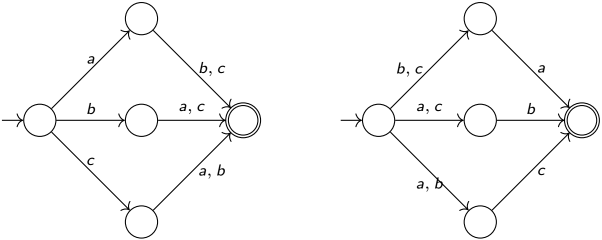
# Minimal DFA

Up to isomorphism, the unique size-minimal DFA accepting
$\mathcal{L} = \{ab, ac, ba, bc, ca, cb\} \subseteq \{a, b, c\}^*$:



# Minimal NFA

Two non-isomorphic[1] size-minimal NFA accepting
$\mathcal{L} = \{ab, ac, ba, bc, ca, cb\} \subseteq \{a, b, c\}^*$:

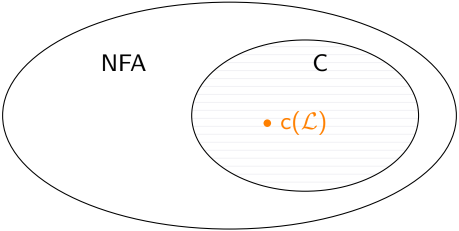



Is there a canonical NFA for $\mathcal{L}$?

[1] Arnold, Dicky, and Nivat 1992.

# Minimal NFA

Is there a subclass $\mathsf{C} \subseteq \mathsf{NFA}$, such that any regular language $\mathcal{L}$ admits a canonical acceptor $\mathsf{c}(\mathcal{L}) \in \mathsf{C}$ size-minimal in $\mathsf{C}$?



# Example: The canonical RFSA

A NFA accepting $\mathcal{L} \subseteq A^*$ is RFSA, if every state accepts a residual $u^{-1}\mathcal{L} = \{v \in A^* \mid uv \in \mathcal{L}\}$ for some $u \in A^*$.

The canonical RFSA for a regular language $\mathcal{L} \subseteq A^*$ is the $X_0$-pointed NFA $\langle \varepsilon, \delta \rangle : X \to 2 \times \mathcal{P}(X)^A$ given by:

- $X = \{U \subseteq A^* \mid U \text{ prime residual of } \mathcal{L}\}$;
- $X_0 = \{U \in X \mid U \subseteq \mathcal{L}\}$;
- $\varepsilon(U) = [\varepsilon \in U]$;
- $\delta_a(U) = \{V \in X \mid V \subseteq a^{-1}U\}$.

## Theorem ([2])

*The canonical RFSA for $\mathcal{L}$ is size-minimal among RFSA for $\mathcal{L}$.*

[2] Denis, Lemay, and Terlutte 2002.

# Example: The canonical RFSA

How does one come up with this definition? Why does it work?

# NFA $\rightarrow$ DFA (in CSL)

The classical powerset construction converts a NFA into an equivalent DFA.

$$\langle \varepsilon, \delta \rangle : Y \to 2 \times \mathcal{P}(Y)^A$$

$$\downarrow$$

$$\langle \varepsilon^\sharp, \delta^\sharp \rangle^3 : \mathcal{P}(Y) \to 2 \times \mathcal{P}(Y)^A$$

[3] $\varepsilon^\sharp(U) = \bigvee_{u \in U} \varepsilon(u), \quad \delta_a^\sharp(U) = \bigcup_{u \in U} \delta_a(u)$

# DFA (in CSL) $\to$ NFA

Consider the reverse to the powerset construction.

$$\begin{array}{c} \langle E, D \rangle : X \to 2 \times X^A \\ \big\downarrow^{4} \\ \langle \varepsilon, \delta \rangle : Y \to 2 \times \mathcal{P}(Y)^A \end{array} \qquad\qquad 2, X \in \mathrm{CSL}$$

Possible? Maybe, choose $Y$ as a generator for $X$? Can we find a size-minimal generator $Y$?

[4] Constraint: $\langle D, E \rangle \sim \langle \delta^\sharp, \varepsilon^\sharp \rangle$

# $T$-DFA $\to$ $T$-NFA

Generalises to other algebraic theories $T$:

$$\begin{array}{c} X \to B \times X^A \\ \big\downarrow \\ Y \to B \times T(Y)^A \end{array} \qquad\qquad B, X \in \mathrm{Alg}(T)$$

Allows the construction of canonical (minimal) automata:

- canonical RFSA[5] (T=CSL, B=2)
- canonical nominal RFSA[6] (T=Nominal CSL, B=2)
- minimal xor automaton[7] ($T=\mathbb{Z}_2$-VSP, B=2)

[5] Denis, Lemay, and Terlutte 2002.

[6] Moerman and Sammartino 2020.

[7] Vuillemin and Gama 2010.

# Example: The átomaton

Previous approach is not general enough to capture e.g. the átomaton[8], which intertwines CABA and CSL.

The átomaton for a regular language $\mathcal{L} \subseteq A^*$ is the $X_0$-pointed NFA $\langle \varepsilon, \delta \rangle : X \to 2 \times \mathcal{P}(X)^A$ given by:

- $X = \{U \subseteq A^* \mid U \text{ atom of } \mathcal{L}\}$;
- $X_0 = \{U \in X \mid U \subseteq \mathcal{L}\}$;
- $\varepsilon(U) = [\varepsilon \in U]$;
- $\delta_a(U) = \{V \in X \mid V \subseteq a^{-1}U\}$.

[8] Brzozowski and Tamm 2014.

# $S$-DFA $\to$ $T$-NFA

Need a situation parametric in two theories $S$, $T$:

$$\begin{array}{c} X \to B \times X^A \\ \downarrow \\ Y \to B \times T(Y)^A \end{array} \qquad B, X \in \mathrm{Alg}(S)$$

Rough idea:

- átomaton (S = CABA, T = CSL, B = 2)
- distromaton[9] (S = CDL, T = CSL, B = 2)
- ...

[9] Myers et al. 2015.

# Contributions

- Categorical framework for the derivation of canonical automata
- Strictly improve expressivity of previous work
- Cover categories different from set, e.g. nominal sets
- Discover a new canonical acceptor by relating mod-2 vector spaces with CABAs
- Present sufficient conditions for the existence of minimal acceptors
- Subsume and establish new minimality results

# Overview

Formally, we make the following generalisations:

| | |
|---|---|
| CSL | $TX \to X \in \mathsf{Alg}(T)$ |
| DFA | $X \to FX \in \mathsf{Coalg}(F)$ |
| CSL-DFA | $TX \to X \to FX \in \mathsf{Bialg}(\lambda)$ |
| CSL-NFA | $T^2Y \to TY \to FTY \in \mathsf{Bialg}(\lambda)$ |

# Generators

A generator[10] for a $T$-algebra $\langle X, h \rangle$ is a tuple $\langle Y, i, d \rangle$ consisting of an object $Y$ and a pair of morphisms

$$TY \underset{d}{\overset{i^\sharp}{\rightleftarrows}} X \qquad \text{satisfying} \qquad i^\sharp \circ d = \mathrm{id}_X,$$

where $i^\sharp := h \circ Ti : TY \to X$ is the unique extension of $i : Y \to X$ to a $T$-algebra homomorphism[11].

If in addition $d \circ i^\sharp = \mathrm{id}_{TY}$, we speak of a basis.

[10] Arbib and Manes 1975.

[11] For instance, every $T$-algebra $\langle X, h \rangle$ is generated by $\langle X, \mathrm{id}_X, \eta_X \rangle$.

# Generators

$\langle Y, i, d\rangle$ is a generator for an algebra $\langle X, h\rangle$ over the powerset monad iff for all $x \in X$

$$x = \bigvee_{y \in d(x)}^{h} i(y).$$

$\langle Y, i, d\rangle$ is a generator for an algebra $\langle X, h\rangle$ over the free mod-2 vector-space monad iff for all $x \in X$

$$x = \bigoplus_{y \in Y}^{h} d(x)(y) \cdot^{h} i(y).$$

# Generators

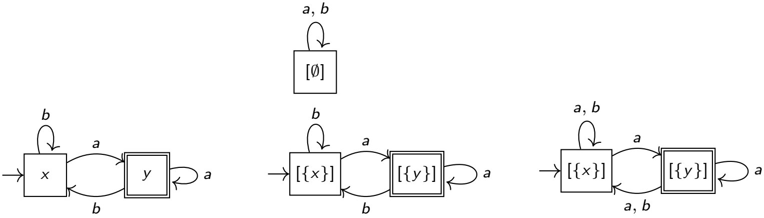
Let $\langle X, h, k\rangle$ be a $\lambda$-bialgebra and $\langle Y, i, d\rangle$ a generator for the $T$-algebra $\langle X, h\rangle$.

**Lemma**

*The morphism $h \circ Ti : TY \to X$ is a $\lambda$-bialgebra homomorphism*

$$h \circ Ti : \langle TY, \mu_Y, (Fd \circ k \circ i)^\sharp\rangle \to \langle X, h, k\rangle.$$

# Example: The canonical RFSA

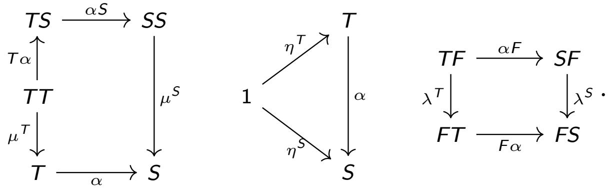



(a) (b) (c)

Figure:
(a) The minimal DFA for $\mathcal{L} = (a + b)^* a$;
(b) The minimal CSL-structured DFA $\langle X, h, k \rangle$ for $\mathcal{L}$;
(c) The canonical RFSA $\langle J(\langle X, h \rangle), Fd \circ k \circ i \rangle$ for $\mathcal{L}$.

# Distributive law homomorphisms

A distributive law homomorphism[12] $\alpha : \lambda^S \to \lambda^T$ between $\lambda^S : SF \Rightarrow FS$ and $\lambda^T : TF \Rightarrow FT$ consists of a natural transformation $\alpha : T \Rightarrow S$ satisfying:




## Lemma ([13])

*Then* $\alpha\langle X, h, k\rangle := \langle X, h \circ \alpha_X, k\rangle$ *and* $\alpha(f) := f$ *defines a functor* $\alpha : \mathrm{Bialg}(\lambda^S) \to \mathrm{Bialg}(\lambda^T)$.

[12] Watanabe 2002; Power and Watanabe 2002.

[13] Klin and Nachyla 2015; Bonsangue et al. 2013.

# Example: The átomaton

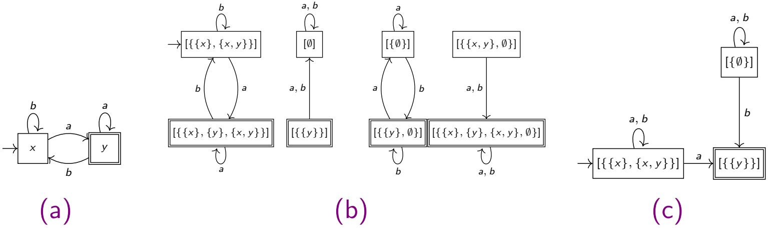
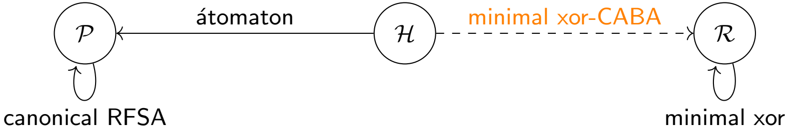



Figure:
(a) The minimal DFA for $\mathcal{L} = (a + b)^* a$;
(b) The minimal CABA-structured DFA $\langle X, h, k \rangle$ for $\mathcal{L}$;
(c) The átomaton $\langle \mathrm{At}(\langle X, h \rangle), Fd \circ k \circ i \rangle$ for $\mathcal{L}$.

# The minimal xor-CABA automaton

"The minimal xor-CABA automaton is to the minimal xor automaton what the átomaton is to the canonical RFSA":



# Minimality

We establish minimality among the following subclasses of $\mathcal{Y}$ accepting $\mathcal{L}$:

Lemma

| | |
|---|---|
| canonical RFSA | $\overline{\mathsf{obs}(\mathcal{Y})}^{\mathsf{CSL}} \subseteq \overline{\mathsf{Der}(\mathcal{L})}^{\mathsf{CSL}}$ |
| minimal xor | all |
| átomaton | $\overline{\mathsf{obs}(\mathcal{Y})}^{\mathsf{CSL}} = \overline{\mathsf{obs}(\mathcal{Y})}^{\mathsf{CABA}}$ |
| distromaton | $\overline{\mathsf{obs}(\mathcal{Y})}^{\mathsf{CSL}} = \overline{\mathsf{obs}(\mathcal{Y})}^{\mathsf{CDL}}$ |
| minimal xor-CABA | $\overline{\mathsf{obs}(\mathcal{Y})}^{\mathbb{Z}_2\text{-}\mathsf{Vect}} = \overline{\mathsf{obs}(\mathcal{Y})}^{\mathsf{CABA}}$ |

# Future work

Some ideas for future work:

- Cover the canonical probabilistic RFSA[14] and canonical alternating RFSA[15];
- Utilise distributive laws between two different categories (e.g. automata product);
- Generalise Brzozowski[16] inspired double reversal characterisations.

[14] Esposito et al. 2002.

[15] Berndt et al. 2017.

[16] Brzozowski 1962.

# The end

Thanks for listening!

[16] https://arxiv.org/abs/2104.13421