Approaches to duality: an overview

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Outline

Introduction

String diagrams

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- The unit object $\mathbf{1}$ is transparent:  : $\mathbf{1} \to A$

Finite-dimensional vector spaces

Example

The monoidal category $(\mathrm{vect}_{\mathbb{k}}, \otimes, \mathbb{k})$ of finite-dimensional vector spaces over a field $\mathbb{k}$ with the usual tensor product.

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-  $: V^\star \otimes V \to \Bbbk, \quad v' \otimes v \mapsto v'(v),$
-  $: \Bbbk \to V \otimes V^\star, \quad 1 \mapsto \sum_i v_i \otimes v_i^\star.$

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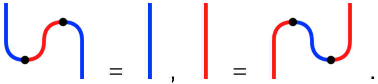
- 

Rigid categories

Definition

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a monoidal category.

- $A \dashv B$:⇔ there exists  $: A \otimes B \to \mathbf{1}$,  $: \mathbf{1} \to B \otimes A$ s.t.

- $(\mathcal{C}, \otimes, \mathbf{1})$ rigid/autonomous :⇔ for all $A \in \mathcal{C}$ there exist ${}^{\vee}A, A^{\vee} \in \mathcal{C}$, s.t. $A \dashv A^{\vee}$ and ${}^{\vee}A \dashv A$.

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However, definition of rigidity is sometimes too restrictive.

Properties of rigid categories

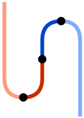
Duality as functor

Definition

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a rigid category and $A, B, A^\vee, B^\vee \in \mathcal{C}$ with

 $: B \otimes B^\vee \to \mathbf{1}$, and  $: \mathbf{1} \to A^\vee \otimes A$.

- For $f \equiv$  $: A \to B$ define

$f^\vee :=$  $: B^\vee \to A^\vee$.

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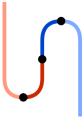
$f^\vee :=$  $: B^\vee \to A^\vee$.

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Lemma

$(-)^\vee : (\mathcal{C}, \otimes, \mathbf{1}) \to (\mathcal{C}, \otimes, \mathbf{1})^{\mathrm{opp}(0,1)}$ *monoid. equiv. with quasi-inv.* ${}^\vee(-)$.

Internal homs

Example

The hom space of $\mathrm{vect}_{\mathbb{k}}$ is an internal hom:

$$\mathrm{Hom}_{\mathbb{k}}(V \otimes W, U) \cong \mathrm{Hom}_{\mathbb{k}}(V, \mathrm{Hom}_{\mathbb{k}}(W, U)).$$

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Definition

- $(\mathcal{C}, \otimes, \mathbf{1})$ left closed :⇔ there exists functor $\underline{\mathrm{Hom}}(-,-) : \mathcal{C}^{\mathrm{opp}} \times \mathcal{C} \to \mathcal{C}$, s.t.

$$\mathrm{Hom}_{\mathcal{C}}(A \otimes B, C) \cong \mathrm{Hom}_{\mathcal{C}}(A, \underline{\mathrm{Hom}}(B, C)).$$

- $(\mathcal{C}, \otimes, \mathbf{1})$ right closed :⇔ there exists functor $\underline{\widetilde{\mathrm{Hom}}}(-,-) : \mathcal{C}^{\mathrm{opp}} \times \mathcal{C} \to \mathcal{C}$, s.t.

$$\mathrm{Hom}_{\mathcal{C}}(A \otimes B, C) \cong \mathrm{Hom}_{\mathcal{C}}(B, \underline{\widetilde{\mathrm{Hom}}}(A, C)).$$

Dualizing objects

Lemma

Any rigid category $(\mathcal{C}, \otimes, \mathbf{1})$ *is biclosed:*

$$\underline{\mathrm{Hom}}(A, B) = B \otimes A^{\vee} \text{ and } \widetilde{\underline{\mathrm{Hom}}}(A, B) = {}^{\vee}A \otimes B.$$

In particular, since $\underline{\mathrm{Hom}}(-, \mathbf{1}) \cong (-)^{\vee}$ *and* $\widetilde{\underline{\mathrm{Hom}}}(-, \mathbf{1}) \cong {}^{\vee}(-)$,

$$\underline{\mathrm{Hom}}(\widetilde{\underline{\mathrm{Hom}}}(A, \mathbf{1}), \mathbf{1}) \cong A \cong \widetilde{\underline{\mathrm{Hom}}}(\underline{\mathrm{Hom}}(A, \mathbf{1}), \mathbf{1}).$$

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Definition

$(\mathcal{C}, \otimes, \mathbf{1})$ biclosed monoidal. Call $k \in \mathcal{C}$ a dualizing object, if

$$\underline{\mathrm{Hom}}(\underline{\widetilde{\mathrm{Hom}}}(A,k),k) \cong A \cong \underline{\widetilde{\mathrm{Hom}}}(\underline{\mathrm{Hom}}(A,k),k)$$

for all $A \in \mathcal{C}$.

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Some more properties...

Since any rigid category is left closed with $\underline{\mathrm{Hom}}(A,B) = B \otimes A^\vee$ and the duality functor is contravariant, we obtain

$$\mathrm{Hom}_{\mathcal{C}}(A \otimes B, C^\vee) \cong \mathrm{Hom}_{\mathcal{C}}(A, C^\vee \otimes B^\vee) \cong \mathrm{Hom}_{\mathcal{C}}(A, (B \otimes C)^\vee).$$

In particular for $C = \mathbf{1}$ we can deduce

$$\mathrm{Hom}(A \otimes B, \mathbf{1}) \cong \mathrm{Hom}(A, B^\vee).$$

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Analogously to the dualizing object definition, we can generalise latter equations. It turns out that all three approaches are equivalent!

Star-autonomous categories

$\star$-autonomous categories

Lemma

Let $(\mathcal{C}, \otimes, \mathbf{1})$ *be a monoidal category. The following are equivalent:*

1. $\mathcal{C}$ *is biclosed and there exists a dualizing object* $k \in \mathcal{C}$*.*
2. *There exists an equivalence* $(-)^\star : \mathcal{C} \to \mathcal{C}^{\mathrm{opp}(1)}$ *and a natural isomorphism*

$$\mathrm{Hom}(A \otimes B, C^\star) \cong \mathrm{Hom}(A, (B \otimes C)^\star).$$

3. *There exists an object* $k \in \mathcal{C}$*, an equivalence* $(-)^\star : \mathcal{C} \to \mathcal{C}^{\mathrm{opp}(1)}$ *and a natural isomorphism*

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Definition

Call such a category $\star$-autonomous (Barr 1979) or Grothendieck-Verdier (Boyarchenko, Drinfeld 2011).

⋆-autonomous category of topological vector spaces

Example

- **TVS** := category of Hausdorff locally convex topological vector spaces with continuous maps.
  $\mathbf{TVS}_m, \mathbf{TVS}_w \subseteq \mathbf{TVS}$ subcategories of weakly, respective Mackey topologized spaces.

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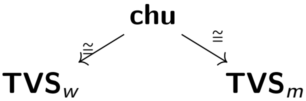
- 

- **chu** is $\star$-autonomous with $(V, W, \langle -,- \rangle)^\star := (W, V, \langle -,- \rangle^{\mathrm{op}})$.

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- $$\mathbf{TVS}_w \xleftarrow{\cong} \mathbf{chu} \xrightarrow{\cong} \mathbf{TVS}_m$$
- **chu** is $\star$-autonomous with $(V, W, \langle -, - \rangle)^\star := (W, V, \langle -, - \rangle^{\mathrm{op}})$.
- Induces tensor product and duality on $\mathbf{TVS}_w$ and $\mathbf{TVS}_m$.

De Morgan's laws

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Important difference between being $\star$-autonomous and being rigid:

$$(-)^{\star} : \mathcal{C} \to \mathcal{C}^{\mathrm{opp}(1)}$$

is not necessarily monoidal in the former case. Thus one can define a second tensor product

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How are the two tensor products related?

Linearly distributive categories

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Definition (Cockett, Seely 1997)

A linearly distributive category is a category $\mathcal{C}$ with two monoidal structures $(\otimes_1, \mathbf{1}_1, \alpha_1, \lambda_1, \rho_1)$, $(\otimes_2, \mathbf{1}_2, \alpha_2, \lambda_2, \rho_2)$, and, not necessarily invertible, natural transformations

$$\delta^L : A \otimes_1 (B \otimes_2 C) \to (A \otimes_1 B) \otimes_2 C,$$
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called distributors, subject to certain pentagon and triangle constraints.

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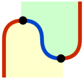
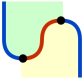
Every monoidal category is linearly distributive with distributors given by the associator and its inverse.

Duality in linearly distributive categories

Definition

Let $\mathcal{C}$ be a linearly distributive category.

- $A \dashv B$:⇔ there exist  $: \mathbf{1}_1 \to B \otimes_2 A$,  $: A \otimes_1 B \to \mathbf{1}_2$, s.t.

 = ,  = .

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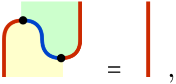
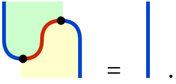

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Properties of linearly distributive categories

Lemma

Let $\mathcal{C}$ be linearly distributive with duality. Then

$$(-)^\star : (\mathcal{C}, \otimes_1, \mathbf{1}_1) \to (\mathcal{C}, \otimes_2, \mathbf{1}_2)^{\mathsf{opp}(0,1)}$$

is a monoidal equivalence with quasi-inverse ${}^\star(-)$.

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In particular $\underline{\mathrm{Hom}}(\underline{\widetilde{\mathrm{Hom}}}(A, \mathbf{1}_2), \mathbf{1}_2) \cong A \cong \underline{\widetilde{\mathrm{Hom}}}(\underline{\mathrm{Hom}}(A, \mathbf{1}_2), \mathbf{1}_2)$.

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- $(G\text{-vect}, \otimes_{g_0}, \mathbb{C}_{g_0}) \overset{(-)^{g_1}}{\cong} (G\text{-vect}, \otimes_{g_1 g_0^{-1}}, \mathbb{C}_{g_1 g_0^{-1}})^{\mathrm{opp}(0,1)}$

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- $(V \otimes_{g_0} W)_g := (V^{g_0} \otimes W^{g_0})^{g_0} \cong (V \otimes W)_{gg_0}$
- $(G\text{-vect}, \otimes_{g_0}, \mathbb{C}_{g_0}) \overset{(-)^{g_1}}{\cong} (G\text{-vect}, \otimes_{g_1 g_0^{-1}}, \mathbb{C}_{g_1 g_0^{-1}})^{\mathrm{opp}(0,1)}$
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Relation to ⋆-autonomous categories

Proposition (Cockett, Seely 1997)

The notion of ⋆-autonomous category coincides with the notion of a linearly distributive category with duality.

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- $q \in \mathrm{QF}(G, \mathbb{C}^\times)$, s.t. $q(g) = q(g_0 g^{-1}) \Rightarrow$ Duality $(-)^{g_0}$ is compatible with braiding and twisting in the ribbon sense.

Frobenius pseudomonoids

Frobenius algebras

Example

R : commutative ring

- $\mathrm{Mat}_n(R)$ is an associative R-algebra, i.e. a monoid in the monoidal category of R-modules.

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This is an example of a Frobenius algebra.

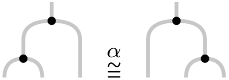
Frobenius algebras can be generalised to Frobenius pseusdomonoids in monoidal bicategories.

Pseudomonoids in bicategories

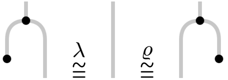
Definition

$(\mathcal{C}, \otimes, \mathbf{1})$ monoidal bicategory.
$(A, \mu, \eta) := (A, \mu, \eta, \alpha, \lambda, \varrho)$ pseudomonoid $:\Leftrightarrow$

- 1-morphisms $\mu \equiv$  , $\eta \equiv$  ,
- invertible 2-morphisms



,



,

- subject to certain pentagon and triangle constraints.

Duality in bicategories

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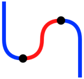
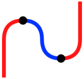
$(\mathcal{C}, \otimes, \mathbf{1})$ monoidal bicategory.

$A \dashv B$:⇔ there exist 1-morphisms  $: A \otimes B \to \mathbf{1}$,  $: \mathbf{1} \to B \otimes A$

and invertible 2-morphisms  $\cong$, $\cong$ 

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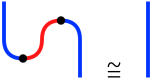
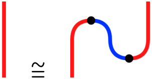
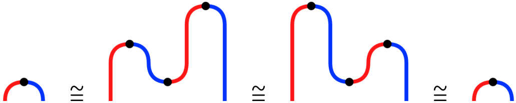
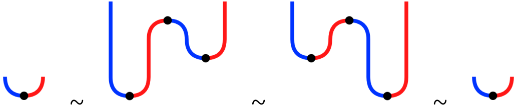
and invertible 2-morphisms  ,  such that

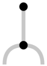
id = (),

id = ().

Frobenius pseudomonoids

Lemma

(A,  *,*  *) pseudomonoid in bicategory $(\mathcal{C}, \otimes, \mathbf{1})$. T.f.a.e.:*

- *There exists*  *, s.t.*  *witnesses $A \dashv A$.*

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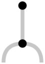
- *There exists*  *, s.t.*  *witnesses $A \dashv A$.*
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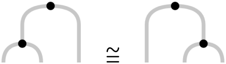

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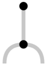
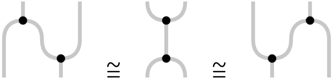
- *There exists  , s.t.  witnesses $A \dashv A$.*
- *There exists  , witnessing $A \dashv A$, and an invertible 2-morphism*

- *There exists pseudocomonoid structure (A,  , ) on A and invertible 2-morphisms*



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The corbordism category Cob(n) has

- objects: closed oriented smooth $(n-1)$-dim. manifolds
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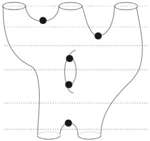
Example

Imagine $\mathrm{Cob}(2)$ partially as  .

Classification of cobordism categories

Lemma

Cob*(2) is the free symmetric monoidal category generated by a commutative Frobenius monoid:*

- *Object:* 
- *Morphisms:* 
- *Relations:*  , ...

Topological field theories

Definition (Atiyah, Segal 1988)

Let $\mathbb{k}$ be a field. A topological field theory of dimension n is a symmetric monoidal functor

$$Z : (\mathrm{Cob}(n), \coprod, \varnothing) \to (\mathrm{vect}_{\mathbb{k}}, \otimes, \mathbb{k}).$$

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Proposition (Folklore, Abrams 1996)

The functor $Z \mapsto Z($  $)$ provides an equivalence between the category of topological field theories in dimension 2 and the category of commutative Frobenius algebras over $\mathbb{k}$.

Relation to $\star$-autonomous categories

Recall: $(\mathcal{C}, \otimes, \mathbf{1})$ $\star$-autonomous :$\Leftrightarrow$ There exists an equivalence $(-)^\star : \mathcal{C} \to \mathcal{C}^{\mathrm{opp}(1)}$ and a natural isomorphism

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However, $\mathbf{Set} \neq \mathbf{1}_{\mathbf{Cat}}$! But we can fix this.

Profunctors

Definition

The monoidal bicategory **Prof** consists of

- objects: small categories,

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- 2-morphisms: natural transformations between functors $\mathcal{D}^{\mathrm{opp}(1)} \times \mathcal{C} \to \mathbf{Set}$.

Frobenius pseudomonoids in **Prof**

Cat embedds into **Prof** in two canonical ways:

- $\mathcal{C} \mapsto \mathcal{C}$
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- $F_\star \dashv F^\star$ *for all* F.
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Proposition

Frobenius pseudomonoid structures of $(\mathcal{C}^{\mathrm{opp}(1)}, (\mu^{\mathrm{opp}(1)})_\star, (\eta^{\mathrm{opp}(1)})_\star) \in$ **Prof** *are in bijection to* $\star$*-autonomous structures of* $(\mathcal{C}, \mu, \eta) \in$ **Cat**.

The end

Thank you!