

Bases for algebras over a monad

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One of the fundamental notions of linear algebra is the concept of a basis for a vector space. In the category theoretic formulation of universal algebra, vector spaces are the Eilenberg-Moore algebras over the free vector space monad on the category of sets. In this paper we show that the notion of a basis can be extended to algebras of arbitrary monads on arbitrary categories. On the one hand, we establish purely algebraic results, for instance about the existence and uniqueness of bases and the representation of algebra morphisms. On the other hand, we use the general notion of a basis in the context of coalgebraic systems and show that a basis for the underlying algebra of a bialgebra gives rise to an equivalent free bialgebra. As a result, we are, for instance, able to recover known constructions from automata theory, namely the so-called canonical residual finite state automaton. Finally, we instantiate the framework to a variety of example monads, including the powerset, downset, distribution, and neighbourhood monad.

1 Introduction

One of the central concepts of linear algebra is the notion of a basis for a vector space: a subset of a vector space is called a basis for the former if every vector can be uniquely written as a finite linear combination of basis elements. Part of the importance of bases stems from the convenient consequences that follow from their existence. For example, linear transformations between vector spaces admit matrix representations relative to pairs of bases [1], which can be used for efficient numerical calculations. The idea of a basis however is not restricted to the theory of vector spaces: other algebraic theories have analogous notions of bases – sometimes by waiving the uniqueness constraint –, for instance modules, semi-lattices, Boolean algebras, convex sets, and many more. In fact, the theory of bases for vector spaces is different to others only in the sense that every vector space admits a basis, which is not the case for e.g. modules. In this paper we seek to give a compact definition of a basis that subsumes the well-known cases, and as a consequence allows us to lift results from one theory to the others. For example, one may wonder if there exists a matrix representation theory for convex sets that is analogous to the one of vector spaces.

In the category theoretic approach to universal algebra, algebraic structures are typically captured as algebras over a monad [2, 3]. Intuitively, a monad may be seen as a generalisation of closure operators on partially ordered sets, and an algebra over a monad may be viewed as a set with an operation that allows the interpretation of formal linear combinations in a way that is coherent with the monad structure. For instance, a vector space over a field k , that is, an algebra for the free k -vector space monad, is given by a set X with a function h that coherently interprets a finitely supported X -indexed k -sequence λ as an actual linear combination $h(\lambda) = \sum_x \lambda_x \cdot x$ in X [4]. It is straightforward to see that under this perspective a basis for a vector space thus consists of a subset Y of X and a function d that assigns to a vector x in X a Y -indexed k -sequence $d(x)$ such that $h(d(x)) = x$ for all x in X and $d(h(\lambda)) = \lambda$ for all Y -indexed k -sequences λ . In other words, the restriction of h to Y -indexed k -sequences is an isomorphism with inverse d , and surjectivity corresponds to the fact that the subset Y generates the vector space, while injectivity captures that Y does so uniquely. As demonstrated in Definition 4.1, the concept easily generalises to arbitrary monads on arbitrary categories by making the subset relation explicit in form of a function.

Monads however not only occur in the context of universal algebra, but also play a role in algebraic topology [5] and theoretical computer science [6, 7, 8]. Among others, they are a convenient tool for capturing side-effects of coalgebraic systems [9]: popular examples include the powerset monad (non-determinism), the distribution monad (probability), and the neighbourhood monad (alternating). While coalgebraic systems with side-effects can be more compact than their deterministic counterparts, they often lack a unique minimal acceptor for a given language. For instance, every regular language admits a unique deterministic automaton with minimal state space, which can be computed via the Myhill-Nerode construction. On the other hand, for some regular languages there exist multiple non-isomorphic non-deterministic automata with minimal state space. The problem has been independently approached for different variants of side-effects, often with the common idea of restricting to a particular subclass [10, 11]. For example, for non-deterministic automata, the subclass of so-called residual finite state automata has been identified as suitable. It moreover has turned out that in order to construct a unique minimal realisation in one of the subclasses it is often sufficient to derive an equivalent system with free state space from a particular given system [12]. As Arbib and Manes realised [13], instrumental to the former is what they call a *scoop*, or what we call a generator in Definition 3.1, a slight generalisation of bases. In other words, our definition of a basis for an algebra over a monad has its origin in a context that is not purely concerned with universal algebra. Throughout the paper we will value these roots by lifting results of Arbib and Manes from scoops to bases. More importantly, we believe that our treatment allows us to uncover hidden ramifications between certain areas of universal algebra and the theory of coalgebras.

The paper is structured as follows. In Section 2 we recall the basic categorical notions of a monad, algebras over a monad, coalgebras, distributive laws, and bialgebras. In Section 3 we introduce generators for algebras over monads and exemplify their relation with free bialgebras. The definition of bases for algebras over monads, basic results, and their relation with free bialgebras is covered in Section 4. Questions about the existence and the uniqueness of bases are answered in Section 4.1 and Section 4.2, respectively. In Section 4.3 we generalise the representation theory of linear maps between vector spaces to a representation theory of homomorphisms between algebras over a monad. The intuition that bases for an algebra over a monad coincide with free isomorphic algebras is clarified in Section 4.4. In Section 4.5 we look into bases for bialgebras, which are algebras for a particular monad. In Section 5 we instantiate the theory for a variety of monads. Related work and future work are discussed in Section 6 and Section 7, respectively. Further details can be found in Appendix A.

2 Preliminaries

We only assume a basic knowledge of category theory, e.g. an understanding of categories, functors, natural transformations, and adjunctions. All other relevant definitions can be found in the paper. In this section, we recall the notions of a monad, algebras for a monad, coalgebras, distributive laws, and bialgebras.

The concept of a monad can be traced back both to algebraic topology [5] and to an alternative to Lawvere theory as a category theoretic formulation of universal algebra [2, 3]. For an extended historical overview we refer to the survey of Hyland and Power [14]. In the context of computer science, monads have been introduced by Moggi as a general perspective on exceptions, side-effects, non-determinism, and continuations [6, 7, 8].

Definition 2.1 (Monad). A *monad* on \mathcal{C} is a tuple $\mathbb{T} = (T, \mu, \eta)$ consisting of an endofunctor $T : \mathcal{C} \rightarrow \mathcal{C}$ and natural transformations $\mu : T^2 \Rightarrow T$ and $\eta : \text{id}_{\mathcal{C}} \Rightarrow T$ satisfying the commutative diagrams

$$\begin{array}{ccc}
 T^3 X & \xrightarrow{T\mu_X} & T^2 X \\
 \mu_{TX} \downarrow & & \downarrow \mu_X \\
 T^2 X & \xrightarrow{\mu_X} & TX
 \end{array}
 \qquad
 \begin{array}{ccc}
 TX & \xrightarrow{\eta_{TX}} & T^2 X \\
 T\eta_X \downarrow & \searrow \text{id}_{TX} & \downarrow \mu_X \\
 T^2 X & \xrightarrow{\mu_X} & TX
 \end{array}$$

for all objects X in \mathcal{C} .

Many examples of monads arise as the result of a free-forgetful adjunction, for instance the free group monad or the free vector space monad. Below we provide some details for the latter case. More monads are covered in Example 4.1 and Section 5.

Example 2.1 (Vector spaces). The free k -vector space monad is an instance of the so-called multiset monad over some semiring S , when S is given by the field k . The underlying set endofunctor T assigns to a set X the set of finitely-supported X -indexed sequences φ in S , typically written as formal sums $\sum_i s_i \cdot x_i$ for $s_i = \varphi(x_i)$; the unit η_X maps an element in X to the singleton multiset $1 \cdot x$; and the multiplication μ_X satisfies $\mu_X(\sum_i s_i \cdot \varphi_i)(x) = \sum_i s_i \cdot \varphi_i(x)$ [4].

If a monad results from a free-forgetful adjunction induced by some algebraic structure, the latter may be recovered in the following sense:

Definition 2.2 (\mathbb{T} -algebra). An *algebra* for a monad $\mathbb{T} = (T, \mu, \eta)$ on \mathcal{C} is a tuple (X, h) consisting of an object X and a morphism $h : TX \rightarrow X$ such that the diagrams on the left and in the middle below commute

$$\begin{array}{ccc}
 T^2X & \xrightarrow{\mu_X} & TX \\
 Th \downarrow & & \downarrow h \\
 TX & \xrightarrow{h} & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 X & \xrightarrow{\text{id}_X} & X \\
 \eta_X \downarrow & \nearrow h & \\
 TX & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 TX & \xrightarrow{Tf} & TY \\
 h_X \downarrow & & \downarrow h_Y \\
 X & \xrightarrow{f} & Y
 \end{array}$$

A *homomorphism* $f : (X, h_X) \rightarrow (Y, h_Y)$ between \mathbb{T} -algebras is a morphism $f : X \rightarrow Y$ such that the diagram on the right above commutes. The category of \mathbb{T} -algebras and homomorphisms is denoted by $\mathcal{C}^{\mathbb{T}}$.

The canonical example for an algebra over a monad is the free \mathbb{T} -algebra (TX, μ_X) for any object X in \mathcal{C} . Below we give some more details on how to recognise algebras over the free vector space monad as vector spaces.

Example 2.2 (Vector spaces). Let \mathbb{T} be the free vector space monad defined in Example 2.1. Every \mathbb{T} -algebra (X, h) induces a vector space structure on its underlying set by interpreting a finite formal linear combination $\sum_i \lambda_i \cdot x_i$ as an element $h(\varphi) \in X$ for $\varphi(x) := \lambda_i$, if $x = x_i$, and $\varphi(x) := 0$ otherwise. Conversely, every vector space with underlying set X induces an algebra (X, h) over \mathbb{T} by defining $h(\varphi) := \sum_x \lambda_x \cdot x$ with $\lambda_x := \varphi(x)$ for a finitely-supported sequence φ .

We now turn our attention to the dual of algebras: coalgebras [9]. While algebras have been used in the context of computer science to model finite data types, coalgebras deal with infinite data types and have turned out to be suited as an abstraction for a variety of state-based systems [15].

Definition 2.3 (F -coalgebra). A *coalgebra* for an endofunctor $F : \mathcal{C} \rightarrow \mathcal{C}$ is a tuple (X, k) consisting of an object X and a morphism $k : X \rightarrow FX$. A *homomorphism* $f : (X, k_X) \rightarrow (Y, k_Y)$ between F -coalgebras is a morphism $f : X \rightarrow Y$ satisfying the commutative diagram

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 k_X \downarrow & & \downarrow k_Y \\
 FX & \xrightarrow{Ff} & FY
 \end{array}$$

The category of coalgebras and homomorphisms is denoted by $\text{Coalg}(F)$. A F -coalgebra (Θ, k_Θ) is *final*, if it is final in $\text{Coalg}(F)$, that is, for every F -coalgebra (X, k_X) there exists a unique homomorphism

$$!_{(X, k_X)} : (X, k_X) \longrightarrow (\Theta, k_\Theta).$$

For example, coalgebras for the set endofunctor $FX = X^A \times B$ are unpointed Moore automata with input A and output B , and the final F -coalgebra homomorphism assigns to a state x of an

unpointed Moore automaton the language in $A^* \rightarrow B$ accepted by the latter when given the initial state x .

We are particularly interested in systems with side-effects, for instance non-deterministic automata. Often such systems can be realised as coalgebras for an endofunctor composed of a monad and an endofunctor similar to F above. In these cases the compatibility between the dynamics of the system and its side-effects can be captured by a so-called a distributive law. Distributive laws have originally occurred as a way to compose monads [16], but now also exist in a wide range of other forms [17]. For our particular case it is sufficient to consider distributive laws between a monad and an endofunctor.

Definition 2.4 (Distributive law). Let $\mathbb{T} = (T, \mu, \eta)$ be a monad on \mathcal{C} and $F : \mathcal{C} \rightarrow \mathcal{C}$ an endofunctor. A natural transformation $\lambda : TF \Rightarrow FT$ is called *distributive law*, if it satisfies

$$\begin{array}{ccc} FX & \xrightarrow{F\eta_X} & FTX \\ \eta_{FX} \downarrow & \nearrow \lambda_X & \\ TFX & & \end{array} \quad \begin{array}{ccccc} T^2FX & \xrightarrow{T\lambda_X} & TFTX & \xrightarrow{\lambda_{TX}} & FT^2X \\ \mu_{FX} \downarrow & & & & \downarrow F\mu_X \\ TFX & \xrightarrow{\lambda_X} & & & FTX \end{array}$$

Given a distributive law, it is straightforward to model the determinisation of a system with side-effects. Indeed, for any morphism $f : Y \rightarrow X$ and \mathbb{T} -algebra (X, h) , the natural isomorphism underlying the free-algebra adjunction yields a \mathbb{T} -algebra homomorphism

$$f^\# := h \circ Tf : (TY, \mu_Y) \longrightarrow (X, h). \quad (1)$$

Thus, in particular, any FT -coalgebra $k : X \rightarrow FTX$ lifts to a F -coalgebra

$$k^\# := (F\mu_X \circ \lambda_{TX}) \circ Tk : (TX, \mu_X) \longrightarrow (FTX, F\mu_X \circ \lambda_{TX}). \quad (2)$$

For instance, if \mathbb{P} is the powerset monad and F is the set endofunctor for deterministic automata satisfying $FX = X^A \times 2$, the disjunctive \mathbb{P} -algebra structure on the set 2 induces a canonical distributive law [18], such that the lifting (2) is given by the classical determinisation procedure for non-deterministic automata [19].

One can show that the state spaces of F -coalgebras obtained by the lifting (2) can canonically be equipped with a \mathbb{T} -algebra structure that is compatible with the F -coalgebra structure: they are λ -bialgebras.

Definition 2.5 (λ -bialgebra). Let λ be a distributive law between a monad \mathbb{T} and an endofunctor F . A λ -bialgebra is a tuple (X, h, k) consisting of a \mathbb{T} -algebra (X, h) and a F -coalgebra (X, k) , satisfying

$$\begin{array}{ccc} TX & \xrightarrow{Tk} & TFX \\ \downarrow h & & \downarrow \lambda_X \\ & & FTX \\ & & \downarrow Fh \\ X & \xrightarrow{k} & FX. \end{array}$$

A *homomorphism* $f : (X, h_X, k_X) \rightarrow (Y, h_Y, k_Y)$ between λ -bialgebras is a morphism $f : X \rightarrow Y$ that is simultaneously a \mathbb{T} -algebra homomorphism and a F -coalgebra homomorphism. The category of λ -bialgebras and morphisms is denoted by $\mathbf{Bialg}(\lambda)$.

It is well-known that a distributive law λ between a monad \mathbb{T} and an endofunctor F induces simultaneously

- a monad $\mathbb{T}_\lambda = (T_\lambda, \mu, \eta)$ on $\mathbf{Coalg}(F)$ by $T_\lambda(X, k) = (TX, \lambda_X \circ Tk)$ and $T_\lambda f = Tf$; and
- an endofunctor F_λ on $\mathcal{C}^\mathbb{T}$ by $F_\lambda(X, h) = (FX, Fh \circ \lambda_X)$ and $F_\lambda f = Ff$,

such that the algebras over \mathbb{T}_λ , the coalgebras of F_λ , and λ -bialgebras coincide [20]. In light of the latter we will not distinguish between the different categories, and instead use the notation of λ -bialgebras for all three cases.

One can further show that, if it exists, the final F -coalgebra (Θ, k_Θ) induces a final F_λ -coalgebra $(\Theta, h_\Theta, k_\Theta)$ for $h_\Theta := !_{(T\Theta, \lambda_\Theta \circ Tk_\Theta)}$ the unique F -coalgebra homomorphism below:

$$\begin{array}{ccc} T\Theta & \overset{h_\Theta}{\dashrightarrow} & \Theta \\ \lambda_\Theta \circ Tk_\Theta \downarrow & & \downarrow k_\Theta \\ FT\Theta & \overset{Fh_\Theta}{\dashrightarrow} & F\Theta \end{array}$$

For instance, for the canonical distributive law between the powerset monad \mathbb{P} and the set endofunctor F with $FX = X^A \times 2$ as before, one verifies that the underlying state space $A^* \rightarrow 2$ of the final F -coalgebra will in this way be enriched with a \mathbb{P} -algebra structure that takes the union of languages [18].

3 Generators for algebras

In this section we define what it means to be a generator for an algebra over a monad. Our notion coincides with what is called a *scoop* by Arbib and Manes [13]. One may argue that the morphism i in the definition below should be mono, but we choose to continue without this requirement.

Definition 3.1 (Generator [13]). A *generator* for a \mathbb{T} -algebra (X, h) is a tuple (Y, i, d) consisting of an object Y , a morphism $i : Y \rightarrow X$, and a morphism $d : X \rightarrow TY$, such that $i^\# \circ d = \text{id}_X$, that is, the diagram on the left below commutes

$$\begin{array}{ccc} TY & \xrightarrow{Ti} & TX \\ \uparrow d & & \downarrow h \\ X & \xrightarrow{\text{id}_X} & X \end{array} \quad \begin{array}{ccc} & TY_\alpha & \\ d_\alpha \nearrow & \downarrow Tf & \\ X & & TY_\beta \\ d_\beta \searrow & & \end{array} \quad \begin{array}{ccc} Y_\alpha & & \\ \downarrow f & \searrow i_\alpha & \\ Y_\beta & & X \\ \nearrow i_\beta & & \end{array}$$

A morphism $f : (Y_\alpha, i_\alpha, d_\alpha) \rightarrow (Y_\beta, i_\beta, d_\beta)$ between generators for (X, h) is a morphism $f : Y_\alpha \rightarrow Y_\beta$ satisfying the two commutative diagrams on the right above.

We give an example that slightly generalises the vector space situation mentioned in the introduction.

Example 3.1 (Semimodules). Let $\mathbb{T} = (T, \mu, \eta)$ be the multiset monad over some semiring S defined in Example 2.1. Following the lines of Example 2.2, one can show that \mathbb{T} -algebras correspond to semimodules over S , such that a function $i : Y \rightarrow X$ is part of a generator (Y, i, d) for the former if and only if for all x in X there exists a finitely-supported Y -indexed S -sequence $d(x) = (s_y)_{y \in Y}$, such that $x = \sum_{y \in Y} s_y \cdot i(y)$.

One might would like to adapt Definition 3.1 by replacing the existence of a morphism d with the property of $i^\#$ being an epimorphism. Indeed, in every category a morphism with a right-inverse is an epimorphism. However, conversely, not every epimorphism admits a right-inverse, and if there would exist one, it might not be unique. For this reason we treat the morphism d as explicit data.

It is well-known that every algebra over a monad admits a generator [13].

Lemma 3.1. (X, id_X, η_X) is a generator for any \mathbb{T} -algebra (X, h) .

Proof. Follows immediately from the equality $h \circ \eta_X = \text{id}_X$. □

The following result is a slight generalisation of a statement by Arbib and Manes [13]. We are particularly interested in using the construction of an equivalent free bialgebra for a unified view on the theory of residual finite state automata and variations of it [21, 22, 23, 24]; more details are given in Example 3.2.

Proposition 3.1. Let (X, h, k) be a λ -bialgebra and let (Y, i, d) be a generator for the \mathbb{T} -algebra (X, h) . Then $i^\sharp := h \circ Ti : TY \rightarrow X$ is a λ -bialgebra homomorphism $i^\sharp : (TY, \mu_Y, (Fd \circ k \circ i)^\sharp) \rightarrow (X, h, k)$ for $(Fd \circ k \circ i)^\sharp := F\mu_Y \circ \lambda_{TY} \circ T(Fd \circ k \circ i)$.

Proof. The commuting diagram below shows that for a FT -coalgebra $f : Y \rightarrow FTY$ and $f^\sharp : TY \rightarrow FTY$ the lifting in (2), the tuple (TY, μ_Y, f^\sharp) constitutes a λ -bialgebra

$$\begin{array}{ccccccc}
T^2Y & \xrightarrow{T^2f} & T^2FTY & \xrightarrow{T\lambda_{TY}} & TFT^2Y & \xrightarrow{TF\mu_Y} & TFTY \\
\downarrow \mu_Y & & \downarrow \mu_{FTY} & & \downarrow \lambda_{T^2Y} & & \downarrow \lambda_{TY} \\
& & & & FT^3Y & \xrightarrow{FT\mu_Y} & FT^2Y \\
& & & & \downarrow F\mu_{TY} & & \downarrow F\mu_Y \\
TY & \xrightarrow{Tf} & TFTY & \xrightarrow{\lambda_{TY}} & FT^2Y & \xrightarrow{F\mu_Y} & FTY
\end{array} \quad (3)$$

In particular, the tuple $(TY, \mu_Y, (Fd \circ k \circ i)^\sharp)$ thus is a λ -bialgebra. The lifting in (1) turns i^\sharp into a \mathbb{T} -algebra homomorphism. It thus remains to show that i^\sharp is a F -coalgebra homomorphism. The latter follows from the commutativity of the following diagram

$$\begin{array}{ccccccc}
TY & \xrightarrow{Ti} & TX & \xrightarrow{h} & X & & \\
\downarrow Ti & & \downarrow Tk & & \downarrow k & & \\
TX & & TX & & & & \\
\downarrow Tk & & \downarrow TFd & & \downarrow TFh & & \\
TFX & \xrightarrow{\text{id}_{TFX}} & TFX & & & & \\
\downarrow TFd & & \downarrow TFh & & \downarrow \lambda_X & & \\
TFTY & \xrightarrow{TFTi} & TFTX & & & & \\
\downarrow \lambda_{TY} & & \downarrow FT^2i & & \downarrow FT^2h & & \\
FT^2Y & \xrightarrow{FT^2i} & FT^2X & \xrightarrow{FT^2h} & FTX & & \\
\downarrow F\mu_Y & & \downarrow F\mu_X & & \downarrow Fh & & \\
FTY & \xrightarrow{FTi} & FTX & \xrightarrow{Fh} & FX & \xrightarrow{\text{id}_{FX}} & FX
\end{array}$$

□

We show next that above construction extends to morphisms $f : (Y_\alpha, i_\alpha, d_\alpha) \rightarrow (Y_\beta, i_\beta, d_\beta)$ between generators for the underlying algebra of a bialgebra. For readability we abbreviate $k_\gamma := (Fd_\gamma \circ k \circ i_\gamma)^\sharp$ for $\gamma \in \{\alpha, \beta\}$.

Lemma 3.2. The morphism $Tf : TY_\alpha \rightarrow TY_\beta$ is a λ -bialgebra homomorphism $Tf : (TY_\alpha, \mu_{Y_\alpha}, k_\alpha) \rightarrow (TY_\beta, \mu_{Y_\beta}, k_\beta)$ satisfying $(i_\beta)^\sharp \circ Tf = (i_\alpha)^\sharp$.

Proof. The identity $(i_\beta)^\sharp \circ Tf = (i_\alpha)^\sharp$ follows from the equality $i_\beta \circ f = i_\alpha$, as shown below

$$\begin{array}{ccccc}
TY_\alpha & \xrightarrow{Tf} & TY_\beta & \xrightarrow{Ti_\beta} & TX \\
\downarrow Ti_\alpha & & \downarrow Ti_\beta & & \downarrow h \\
TX & \xrightarrow{\text{id}_{TX}} & TX & \xrightarrow{h} & X
\end{array}$$

One easily verifies that the lifting (2) extends to a functor from the category of FT -coalgebras to the category of λ -bialgebras; see e.g. (3). It thus remains to show that f is a FT -coalgebra homomorphism $f : (Y_\alpha, k_\alpha) \rightarrow (Y_\beta, k_\beta)$. The latter follows from the commutativity of the following diagram

$$\begin{array}{ccccccc}
Y_\beta & \xrightarrow{i_\beta} & X & \xrightarrow{k} & FX & \xrightarrow{Fd_\beta} & FTY_\beta \\
f \uparrow & & \text{id}_X \uparrow & & \text{id}_{FX} \uparrow & & \uparrow FTf \\
Y_\alpha & \xrightarrow{i_\alpha} & X & \xrightarrow{k} & FX & \xrightarrow{Fd_\alpha} & FTY_\alpha
\end{array}$$

□

In the following example we instantiate the previous results to recover the so-called canonical residual finite state automata [24].

Example 3.2 (Canonical RFSA). As before, let \mathbb{P} be the powerset monad and F the set endofunctor for deterministic automata over the alphabet A satisfying $FX = X^A \times 2$. One verifies that the disjunctive \mathbb{P} -algebra structure on the set 2 induces a canonical distributive law λ between \mathbb{P} and F , such that λ -bialgebras are deterministic unpointed automata in the category of complete lattices and join-preserving functions [18]; for more details see Section 5.1. We are particularly interested in the λ -bialgebra that is typically called the minimal \mathbb{P} -automaton $M_{\mathbb{P}}(L)$ for a regular language L [22, 12]. On a very high-level, $M_{\mathbb{P}}(L)$ may be recognised as some algebraic closure of the well-known minimal automaton $M(L)$ for L in the category of sets and functions. More concretely, it consists of the inclusion-ordered free complete lattice of unions of residuals of L , equipped with the usual transition and output functions for languages inherited by the final coalgebra for F . Using well-known lattice theoretic arguments one can show that the tuple $(\mathcal{J}(M_{\mathbb{P}}(L)), i, d)$, with i the subset-embedding of join-irreducibles and d the function assigning to a language the join-irreducible languages below it, is a generator for $M_{\mathbb{P}}(L)$. Writing k for the F -coalgebra structure of $M_{\mathbb{P}}(L)$, it is easy to verify that the $F\mathbb{P}$ -coalgebra structure $Fd \circ k \circ i$ on $\mathcal{J}(M_{\mathbb{P}}(L))$ mentioned in Proposition 3.1 corresponds precisely to the so-called canonical residual finite state automaton for L [24].

We close this section with a more compact characterisation of generators for free algebras.

Lemma 3.3. *Let $i : Y \rightarrow X$ and $d : X \rightarrow TY$ be morphisms such that the following diagram commutes:*

$$\begin{array}{ccc}
T^2Y & \xrightarrow{T^2i} & T^2X \\
Td \uparrow & & \downarrow \mu_X \\
TX & \xrightarrow{\text{id}_{TX}} & TX
\end{array}$$

Then $(Y, \eta_X \circ i, \mu_Y \circ Td)$ is a generator for the \mathbb{T} -algebra (TX, μ_X) .

Proof. Follows from the commutativity of the diagram below

$$\begin{array}{ccccc}
TY & \xrightarrow{Ti} & TX & \xrightarrow{T\eta_X} & T^2X \\
\mu_Y \uparrow & & \mu_X \uparrow & \searrow \text{id}_{TX} & \downarrow \mu_X \\
T^2Y & \xrightarrow{T^2i} & T^2X & \xrightarrow{\mu_X} & TX \\
Td \uparrow & & & & \downarrow \text{id}_{TX} \\
TX & \xrightarrow{\text{id}_{TX}} & TX & & TX
\end{array}$$

□

4 Bases for algebras

In the last section we adopted the notion of a scoop by Arbib and Manes [13] by introducing generators for algebras over a monad. In this section we extend the former to the definition of a basis for an algebra over a monad by adding a uniqueness constraint. While scoops have mainly occurred in the context of state-based systems, our extension allows us to emphasise their ramifications with universal algebra.

Definition 4.1 (Basis). A *basis* for a \mathbb{T} -algebra (X, h) is a tuple (Y, i, d) consisting of an object Y , a morphism $i : Y \rightarrow X$, and a morphism $d : X \rightarrow TY$, such that $i^\sharp \circ d = \text{id}_X$ and $d \circ i^\sharp = \text{id}_{TY}$, that is, the following two diagrams commute:

$$\begin{array}{ccc} TY & \xrightarrow{Ti} & TX \\ \uparrow d & & \downarrow h \\ X & \xrightarrow{\text{id}_X} & X \end{array} \quad \begin{array}{ccc} TX & \xrightarrow{h} & X \\ \uparrow Ti & & \downarrow d \\ TY & \xrightarrow{\text{id}_{TY}} & TY \end{array} .$$

A *homomorphism* between two bases for (X, h) is a morphism between the underlying generators. The category consisting of bases for a \mathbb{T} -algebra (X, h) and homomorphisms between them is denoted by $\text{Bases}(X, h)$.

We begin with an example for a basis in above sense for the theory of monoids.

Example 4.1 (Monoids). Let $\mathbb{T} = (T, \mu, \eta)$ be the set monad whose underlying endofunctor T assigns to a set X the set of all finite words over the alphabet X ; whose unit η_X assigns to a character in X the corresponding word of length one; and whose multiplication μ_X syntactically flattens words over words over the alphabet X in the usual way. The monad \mathbb{T} is also known as the list monad. One verifies that the constraints for its algebras correspond to the unitality and associativity laws of monoids. A function $i : Y \rightarrow X$ is thus part of a basis (Y, i, d) for a \mathbb{T} -algebra with underlying set X if and only if for all $x \in X$ there exists a unique word $d(x) = [y_1, \dots, y_n]$ over the alphabet Y satisfying $x = i(y_1) \cdot \dots \cdot i(y_n)$.

The next result establishes that the morphism d is in fact an algebra homomorphism, and, intuitively, that elements of a basis are uniquely generated by their image under the monad unit, that is, typically by themselves.

Lemma 4.1. *Let (Y, i, d) be a basis for a \mathbb{T} -algebra (X, h) . Then the following two diagrams commute:*

$$\begin{array}{ccc} TX & \xrightarrow{Td} & T^2Y \\ \downarrow h & & \downarrow \mu_Y \\ X & \xrightarrow{d} & TY \end{array} \quad \begin{array}{ccc} Y & \xrightarrow{i} & X \\ \downarrow \eta_Y & \swarrow d & \\ TY & & \end{array} .$$

Proof. Follows from the commutativity of the following two diagrams

$$\begin{array}{ccc} TX & \xrightarrow{Td} & T^2Y \\ \downarrow \text{id}_{TX} & & \swarrow T^2i \\ TX & \xleftarrow{Th} & T^2X \\ \downarrow h & & \downarrow \mu_X \\ TX & \xleftarrow{Ti} & TY \\ \downarrow h & & \downarrow \text{id}_{TY} \\ X & \xrightarrow{d} & TY \end{array} \quad \begin{array}{ccc} Y & \xrightarrow{i} & X \\ \downarrow \eta_Y & & \downarrow \eta_X \\ TY & \xrightarrow{Ti} & TX \\ \downarrow \text{id}_{TY} & & \swarrow d \\ TY & & X \end{array} \quad \begin{array}{ccc} & & X \\ & \swarrow h & \\ & & \downarrow \text{id}_X \\ & & X \end{array} .$$

□

In consequence we can derive the following three corollaries. First, every algebra with a basis is isomorphic to a free algebra.

Corollary 4.1. *Let (Y, i, d) be a basis for a \mathbb{T} -algebra (X, h) . Then $d : X \rightarrow TY$ is a \mathbb{T} -algebra isomorphism $d : (X, h) \rightarrow (TY, \mu_Y)$.*

Proof. By Lemma 4.1 the morphism d is a \mathbb{T} -algebra homomorphism. From general arguments it follows that the lifting $i^\sharp = h \circ Ti$ is a \mathbb{T} -algebra homomorphism in the reverse direction. Since (Y, i, d) is a basis, d and i^\sharp are mutually inverse. \square

Secondly, an algebra with a basis embeds into the free algebra it spans. The monomorphism is fundamental to an alternative approach to bases [25]; for more details see Appendix A.

Corollary 4.2. *Let (Y, i, d) be a basis for a \mathbb{T} -algebra (X, h) . Then $Ti \circ d : X \rightarrow TX$ is a \mathbb{T} -algebra monomorphism $Ti \circ d : (X, h) \rightarrow (TX, \mu_X)$ with left-inverse $h : (TX, \mu_X) \rightarrow (X, h)$.*

Proof. The morphism $Ti \circ d$ is a \mathbb{T} -algebra homomorphism since by Lemma 4.1 the following diagram commutes

$$\begin{array}{ccccc} TX & \xrightarrow{Td} & T^2Y & \xrightarrow{T^2i} & T^2X \\ h \downarrow & & \downarrow \mu_Y & & \downarrow \mu_X \\ X & \xrightarrow{d} & TY & \xrightarrow{Ti} & TX \end{array}$$

The morphism h is a \mathbb{T} -algebra homomorphism since the equality $h \circ \mu_X = h \circ Th$ holds for all algebras over a monad. By the definition of a generator h is a left-inverse to $Ti \circ d$. The morphism $Ti \circ d$ is mono since every morphism with left-inverse is mono. \square

Thirdly, every algebra homomorphism is uniquely determined by its image on a basis.

Corollary 4.3. *Let (Y, i, d) be a basis for a \mathbb{T} -algebra (X, h_X) , and let (Z, h_Z) be another \mathbb{T} -algebra. For every morphism $f : Y \rightarrow Z$ there exists a \mathbb{T} -algebra homomorphism $f^\sharp : (X, h_X) \rightarrow (Z, h_Z)$ satisfying $f^\sharp \circ i = f$.*

Proof. We define a candidate as follows $f^\sharp := h_Z \circ Tf \circ d$. Using Lemma 4.1 we establish the commutativity of the following two diagrams

$$\begin{array}{ccccccc} X & \xrightarrow{d} & TY & \xrightarrow{Tf} & TZ & \xrightarrow{\text{id}_{TZ}} & TZ \\ & \swarrow i & \uparrow \eta_Y & & \uparrow \eta_Z & & \downarrow h_Z \\ & & Y & \xrightarrow{f} & Z & \xrightarrow{\text{id}_Z} & Z \end{array}$$

$$\begin{array}{ccccccc} TX & \xrightarrow{Td} & T^2Y & \xrightarrow{T^2f} & T^2Z & \xrightarrow{Th_Z} & TZ \\ h_X \downarrow & & \downarrow \mu_Y & & \downarrow \mu_Z & & \downarrow h_Z \\ X & \xrightarrow{d} & TY & \xrightarrow{Tf} & TZ & \xrightarrow{h_Z} & Z \end{array}$$

The first diagram proves the identity $f^\sharp \circ i = f$, and the second diagram shows that f^\sharp is a \mathbb{T} -algebra homomorphism. \square

In Corollary 4.1 it was proven that every algebra with a basis is isomorphic to a free algebra. We show next that the statement can be strengthened to bialgebras. In more detail, in Proposition 3.1 we have seen that a generator for the underlying algebra of a bialgebra allows to construct a bialgebra with free state space, such that i^\sharp extends to a bialgebra homomorphism from the latter to the former. As it turns out, for a basis, the homomorphism i^\sharp is in fact an isomorphism.

Proposition 4.1. *Let (X, h, k) be a λ -bialgebra and let (Y, i, d) be a basis for the \mathbb{T} -algebra (X, h) . Then $d : X \rightarrow TY$ is a λ -bialgebra homomorphism $d : (X, h, k) \rightarrow (TY, \mu_Y, (Fd \circ k \circ i)^\sharp)$ for $(Fd \circ k \circ i)^\sharp := F\mu_Y \circ \lambda_{TY} \circ T(Fd \circ k \circ i)$.*

Proof. As before, it follows by general arguments that $(TY, \mu_Y, (Fd \circ k \circ i)^\sharp)$ is a λ -bialgebra. The morphism d is a \mathbb{T} -algebra homomorphism by Lemma 4.1. It thus remains to show that d is a F -coalgebra homomorphism. The former is established by Lemma 4.1, as shown below

$$\begin{array}{ccccccc}
X & \xrightarrow{\quad k \quad} & & & FX & & \\
\downarrow d & \searrow \text{id}_X & & & \nearrow Fh & & \downarrow Fd \cdot \\
& X & \xrightarrow{\quad k \quad} & & FTX & & \\
& \uparrow h & & & \downarrow FTd & & \\
TY & \xrightarrow{Ti} & TX & \xrightarrow{Tk} & TFX & \xrightarrow{TFd} & TFTY & \xrightarrow{\lambda_{TY}} & FT^2Y & \xrightarrow{F\mu_Y} & FTY
\end{array}$$

□

Corollary 4.4. *Let (X, h, k) be a λ -bialgebra and let (Y, i, d) be a basis for the \mathbb{T} -algebra (X, h) . Then the λ -bialgebras (X, h, k) and $(TY, \mu_Y, (Fd \circ k \circ i)^\sharp)$ are isomorphic.*

Proof. By Proposition 4.1 the morphism d is a λ -bialgebra homomorphism of the right type. By Proposition 3.1 the morphism i^\sharp is a λ -bialgebra homomorphism in reverse direction to d . From the definition of a basis it follows that d and i^\sharp are mutually inverse. □

4.1 Existence of bases

While every algebra over a monad admits a generator, cf. Lemma 3.1, the latter is not necessarily true for a basis. In this section we show that one however can safely assume that every *free* algebra admits a basis. We begin with a characterisation of bases for free algebras that is slightly more compact than the one derived directly from the definition.

Lemma 4.2. *Let $i : Y \rightarrow X$ and $d : X \rightarrow TY$ be morphisms such that the following two diagrams commute:*

$$\begin{array}{ccc}
T^2Y & \xrightarrow{T^2i} & T^2X \\
\uparrow Td & & \downarrow \mu_X \\
TX & \xrightarrow{\text{id}_{TX}} & TX
\end{array}
\quad
\begin{array}{ccc}
TX & \xrightarrow{Td} & T^2Y \\
\uparrow Ti & & \downarrow \mu_Y \cdot \\
TY & \xrightarrow{\text{id}_{TY}} & TY
\end{array}$$

Then $(Y, \eta_X \circ i, \mu_Y \circ Td)$ is a basis for the \mathbb{T} -algebra (TX, μ_X) .

Proof. One part of the claim follows from Lemma 3.3. The other part follows from the commutativity of the following diagram

$$\begin{array}{ccc}
T^2X & \xrightarrow{\mu_X} & TX \\
\uparrow T\eta_X & \nearrow \text{id}_{TX} & \downarrow Td \\
TX & \xrightarrow{Td} & T^2Y \\
\uparrow Ti & & \downarrow \mu_Y \\
TY & \xrightarrow{\text{id}_{TY}} & TY
\end{array}$$

□

Corollary 4.5. *(X, η_X, id_X) is a basis for the \mathbb{T} -algebra (TX, μ_X) .*

Proof. Using the equality $\mu_X \circ T\eta_X = \text{id}_{TX}$, the claim follows from Lemma 4.2 with $i = \text{id}_X$ and $d = \eta_X$. □

4.2 Uniqueness of bases

In this section we investigate the uniqueness of bases for algebras over a monad. To begin with, assume (X, h) is an algebra over a monad \mathbb{T} and we are given a fixed morphism $i : Y \rightarrow X$. Then any two morphisms d_α and d_β turning (Y, i, d_α) and (Y, i, d_β) into bases for (X, h) , respectively, are in fact identical:

$$d_\alpha = d_\alpha \circ i^\sharp \circ d_\beta = d_\beta.$$

If the morphism i is not fixed, we have the following slightly weaker result about the uniqueness of bases:

Lemma 4.3. *Let $(Y_\alpha, i_\alpha, d_\alpha)$ and $(Y_\beta, i_\beta, d_\beta)$ be bases for a \mathbb{T} -algebra (X, h) . Then the \mathbb{T} -algebras $(TY_\alpha, \mu_{Y_\alpha})$ and $(TY_\beta, \mu_{Y_\beta})$ are isomorphic.*

Proof. We have a \mathbb{T} -algebra homomorphism

$$d_\beta \circ (i_\alpha)^\sharp : (TY_\alpha, \mu_{Y_\alpha}) \longrightarrow (TY_\beta, \mu_{Y_\beta})$$

since the components $(i_\alpha)^\sharp$ and d_β are \mathbb{T} -algebra homomorphisms by general arguments and Lemma 4.1, respectively. Analogously, by symmetry, the morphism $d_\alpha \circ (i_\beta)^\sharp$ is a \mathbb{T} -algebra homomorphism in the reverse direction. It is easy to verify that the definition of a basis implies that both morphisms are mutually inverse. \square

If a set monad preserves the set cardinality relation in the sense that

$$|Y_\alpha| \neq |Y_\beta| \quad \text{implies} \quad |TY_\alpha| \neq |TY_\beta|,$$

above result in particular shows that any two bases for a fixed algebra have the same cardinality.

4.3 Representation theory

In this section we use our general definition of a basis to derive a representation theory for homomorphisms between algebras over monads that is analogous to the representation theory for linear transformations between vector spaces.

In more detail, recall that a linear transformation $L : V \rightarrow W$ between k -vector spaces with finite bases $\alpha = \{v_1, \dots, v_n\}$ and $\beta = \{w_1, \dots, w_m\}$, respectively, admits a matrix representation $L_{\alpha\beta} \in \text{Mat}_k(m, n)$ with

$$L(v_j) = \sum_i (L_{\alpha\beta})_{i,j} w_i,$$

such that for any vector v in V the coordinate vectors $L(v)_\beta \in k^m$ and $v_\alpha \in k^n$ satisfy the equality

$$L(v)_\beta = L_{\alpha\beta} v_\alpha.$$

A great amount of linear algebra is concerned with finding bases such that the corresponding matrix representation is in an efficient shape, for instance diagonalised. The following definitions generalise the situation by substituting Kleisli morphisms for matrices.

Definition 4.2. Let $\alpha = (Y_\alpha, i_\alpha, d_\alpha)$ and $\beta = (Y_\beta, i_\beta, d_\beta)$ be bases for \mathbb{T} -algebras (X_α, h_α) and (X_β, h_β) , respectively. The *basis representation* of a \mathbb{T} -algebra homomorphism $f : (X_\alpha, h_\alpha) \rightarrow (X_\beta, h_\beta)$ with respect to α and β is the composition

$$f_{\alpha\beta} := Y_\alpha \xrightarrow{i_\alpha} X_\alpha \xrightarrow{f} X_\beta \xrightarrow{d_\beta} TY_\beta. \quad (4)$$

Conversely, the morphism *associated* with a Kleisli morphism $p : Y_\alpha \rightarrow TY_\beta$ with respect to α and β is the composition

$$p^{\alpha\beta} := X_\alpha \xrightarrow{d_\alpha} TY_\alpha \xrightarrow{Tp} T^2Y_\beta \xrightarrow{\mu_{Y_\beta}} TY_\beta \xrightarrow{Ti_\beta} TX_\beta \xrightarrow{h_\beta} X_\beta. \quad (5)$$

The morphism associated with a Kleisli morphism should be understood as the linear transformation between vector spaces induced by some matrix of the right type. The following result confirms this intuition.

Lemma 4.4. (5) is a \mathbb{T} -algebra homomorphism $p^{\alpha\beta} : (X_\alpha, h_\alpha) \rightarrow (X_\beta, h_\beta)$.

Proof. Using Lemma 4.1 we deduce the commutativity of the following diagram

$$\begin{array}{ccccccccccc}
TX_\alpha & \xrightarrow{Td_\alpha} & T^2Y_\alpha & \xrightarrow{T^2p} & T^3Y_\beta & \xrightarrow{T\mu_{Y_\beta}} & T^2Y_\beta & \xrightarrow{T^2i_\beta} & T^2X_\beta & \xrightarrow{Th_\beta} & TX_\beta \\
\downarrow h_\alpha & & \downarrow \mu_{Y_\alpha} & & \downarrow \mu_{TY_\beta} & & \downarrow \mu_{Y_\beta} & & \downarrow \mu_{X_\beta} & & \downarrow h_\beta \\
X_\alpha & \xrightarrow{d_\alpha} & TY_\alpha & \xrightarrow{Tp} & T^2Y_\beta & \xrightarrow{\mu_{Y_\beta}} & TY_\beta & \xrightarrow{Ti_\beta} & TX_\beta & \xrightarrow{h_\beta} & X_\beta
\end{array}$$

□

The following result establishes a generalisation of the observation that for fixed bases, constructing a matrix representation of a linear transformation on the one hand, and associating a linear transformation to a matrix of the right type on the other hand, are mutually inverse operations.

Lemma 4.5. The operations (4) and (5) are mutually inverse.

Proof. The definitions imply

$$\begin{aligned}
(p^{\alpha\beta})_{\alpha\beta} &= d_\beta \circ (h_\beta \circ Ti_\beta \circ \mu_{Y_\beta} \circ Tp \circ d_\alpha) \circ i_\alpha \\
(f_{\alpha\beta})^{\alpha\beta} &= h_\beta \circ Ti_\beta \circ \mu_{Y_\beta} \circ T(d_\beta \circ f \circ i_\alpha) \circ d_\alpha.
\end{aligned}$$

Using Lemma 4.1 we deduce the commutativity of the diagrams below

$$\begin{array}{ccccccc}
Y_\alpha & \xrightarrow{p} & TY_\beta & \xrightarrow{\text{id}_{TY_\beta}} & TY_\beta & & \\
\downarrow i_\alpha & \searrow \eta_{Y_\alpha} & \downarrow \eta_{TY_\beta} & \downarrow \text{id}_{TY_\beta} & \downarrow d_\beta & & \\
X_\alpha & \xrightarrow{d_\alpha} & TY_\alpha & \xrightarrow{Tp} & T^2Y_\beta & \xrightarrow{\mu_{Y_\beta}} & TY_\beta & \xrightarrow{Ti_\beta} & TX_\beta & \xrightarrow{h_\beta} & X_\beta
\end{array}$$

$$\begin{array}{ccccccc}
X_\alpha & \xrightarrow{\text{id}_{X_\alpha}} & X_\alpha & \xrightarrow{f} & X_\beta & \xrightarrow{\text{id}_{X_\beta}} & X_\beta \\
\downarrow Ti_\alpha \circ d_\alpha & \nearrow h_\alpha & & \nearrow h_\beta & \downarrow d_\beta & & \uparrow h_\beta \\
TX_\alpha & \xrightarrow{Tf} & TX_\beta & \xrightarrow{Td_\beta} & T^2Y_\beta & \xrightarrow{\mu_{Y_\beta}} & TY_\beta & \xrightarrow{Ti_\beta} & TX_\beta
\end{array}$$

□

The next result establishes the compositionality of basis representations: the matrix representation of the composition of two linear transformations is given by the multiplication of the matrix representations of the individual linear transformations. On the left side of the following equation we use the usual Kleisli composition.

Lemma 4.6. $g_{\beta\gamma} \cdot f_{\alpha\beta} = (g \circ f)_{\alpha\gamma}$.

Proof. The definitions imply

$$\begin{aligned}
g_{\beta\gamma} \cdot f_{\alpha\beta} &= \mu_{Y_\gamma} \circ T(d_\gamma \circ g \circ i_\beta) \circ d_\beta \circ f \circ i_\alpha \\
(g \circ f)_{\alpha\gamma} &= d_\gamma \circ (g \circ f) \circ i_\alpha.
\end{aligned}$$

We delete common terms and use Lemma 4.1 to deduce the commutativity of the diagram below

$$\begin{array}{ccccccc}
X_\beta & \xrightarrow{d_\beta} & TY_\beta & \xrightarrow{Ti_\beta} & TX_\beta & \xrightarrow{Tg} & TX_\gamma & \xrightarrow{Td_\gamma} & T^2Y_\gamma \\
\downarrow \text{id}_{X_\beta} & & \downarrow h_\beta & \searrow h_\gamma & & & & & \downarrow \mu_{Y_\gamma} \\
X_\beta & \xrightarrow{g} & X_\gamma & \xrightarrow{d_\gamma} & TY_\gamma & & & &
\end{array}$$

□

Similarly to the previous result, the next observation captures the compositionality of the operation that assigns to a Kleisli morphism its associated homomorphism.

Lemma 4.7. $q^{\beta\gamma} \circ p^{\alpha\beta} = (q \cdot p)^{\alpha\gamma}$.

Proof. The definitions imply

$$\begin{aligned} q^{\beta\gamma} \circ p^{\alpha\beta} &= (h_\gamma \circ Ti_\gamma \circ \mu_{Y_\gamma} \circ Tq \circ d_\beta) \circ (h_\beta \circ Ti_\beta \circ \mu_{Y_\beta} \circ Tp \circ d_\alpha) \\ (q \cdot p)^{\alpha\gamma} &= h_\gamma \circ Ti_\gamma \circ \mu_{Y_\gamma} \circ T\mu_{Y_\gamma} \circ T^2q \circ Tp \circ d_\alpha. \end{aligned}$$

By deleting common terms and using the equality $d_\beta \circ h_\beta \circ Ti_\beta = \text{id}_{TY_\beta}$ it is thus sufficient to show

$$\mu_{Y_\gamma} \circ Tq \circ \mu_{Y_\beta} = \mu_{Y_\gamma} \circ T\mu_{Y_\gamma} \circ T^2q.$$

Above equation follows from the commutativity of the diagram below

$$\begin{array}{ccccc} T^2Y_\beta & \xrightarrow{\mu_{Y_\beta}} & TY_\beta & \xrightarrow{Tq} & T^2Y_\gamma \\ \downarrow T^2q & & \nearrow \mu_{TY_\gamma} & & \downarrow \mu_{Y_\gamma} \\ T^3Y_\gamma & \xrightarrow{T\mu_{Y_\gamma}} & T^2Y_\gamma & \xrightarrow{\mu_{Y_\gamma}} & TY_\gamma \end{array}$$

□

At the beginning of this section we recalled the soundness identity

$$L(v)_\beta = L_{\alpha\beta}v_\alpha$$

for the matrix representation $L_{\alpha\beta}$ of a linear transformation L . The next result is a natural generalisation of this statement.

Lemma 4.8. $d_\beta \circ f = f_{\alpha\beta} \cdot d_\alpha$

Proof. The definitions imply

$$f_{\alpha\beta} \cdot d_\alpha = \mu_{Y_\beta} \circ T(d_\beta \circ f \circ i_\alpha) \circ d_\alpha.$$

Using Lemma 4.1 we deduce the commutativity of the diagram below

$$\begin{array}{ccccccc} X_\alpha & \xrightarrow{d_\alpha} & TY_\alpha & \xrightarrow{Ti_\alpha} & TX_\alpha & \xrightarrow{Tf} & TX_\beta & \xrightarrow{Td_\beta} & T^2Y_\beta \\ \text{id}_{X_\alpha} \downarrow & & \nearrow h_\alpha & & \nearrow h_\beta & & & & \downarrow \mu_{Y_\beta} \\ X_\alpha & \xrightarrow{f} & X_\beta & \xrightarrow{d_\beta} & & & & & TY_\beta \end{array}$$

□

Assume we are given bases α, α' and β, β' for \mathbb{T} -algebras (X_α, h_α) and (X_β, h_β) , respectively. The following result makes clear how the two basis representations $f_{\alpha\alpha'}$ and $f_{\beta\beta'}$ are related.

Proposition 4.2. *There exist Kleisli isomorphisms p and q such that $f_{\alpha'\beta'} = q \cdot f_{\alpha\beta} \cdot p$.*

Proof. The Kleisli morphisms p and q and their respective candidates for inverses p^{-1} and q^{-1} are defined below

$$\begin{aligned} p &:= d_\alpha \circ i_{\alpha'} : Y_{\alpha'} \longrightarrow TY_\alpha & q &:= d_{\beta'} \circ i_\beta : Y_\beta \longrightarrow TY_{\beta'} \\ p^{-1} &:= d_{\alpha'} \circ i_\alpha : Y_\alpha \longrightarrow TY_{\alpha'} & q^{-1} &:= d_\beta \circ i_{\beta'} : Y_{\beta'} \longrightarrow TY_\beta. \end{aligned}$$

From Lemma 4.1 it follows that the diagram below commutes

$$\begin{array}{ccccc}
Y_\alpha & \xrightarrow{i_\alpha} & X_\alpha & \xrightarrow{d_{\alpha'}} & TY_{\alpha'} \\
\downarrow \eta_{Y_\alpha} & & \downarrow \text{id}_{X_\alpha} & & \downarrow Ti_{\alpha'} \\
& & X_\alpha & & \\
& \swarrow d_\alpha & & \searrow h_\alpha & \\
TY_\alpha & \xleftarrow{\mu_{Y_\alpha}} & T^2Y_\alpha & \xleftarrow{Td_\alpha} & TX_\alpha
\end{array}$$

This shows that p^{-1} is a Kleisli right-inverse of p . A symmetric version of above diagram shows that p^{-1} is also a Kleisli left-inverse of p . Analogously it follows that q^{-1} is a Kleisli inverse of q .

The definitions further imply the equalities

$$\begin{aligned}
q \cdot f_{\alpha\beta} \cdot p &= \mu_{Y_{\beta'}} \circ T(d_{\beta'} \circ i_\beta) \circ \mu_{Y_\beta} \circ T(d_\beta \circ f \circ i_\alpha) \circ d_\alpha \circ i_{\alpha'} \\
f_{\alpha'\beta'} &= d_{\beta'} \circ f \circ i_{\alpha'}.
\end{aligned}$$

We delete common terms and use Lemma 4.1 to establish the commutativity of the diagram below

$$\begin{array}{ccccccc}
X_\alpha & \xrightarrow{d_\alpha} & TY_\alpha & \xrightarrow{Ti_\alpha} & TX_\alpha & \xrightarrow{Tf} & TX_\beta \\
\downarrow \text{id}_{X_\alpha} & \searrow & & \swarrow h_\alpha & & & \downarrow Td_\beta \\
& & X_\alpha & & & & T^2Y_\beta \\
& & \downarrow f & & \swarrow h_\beta & & \downarrow \mu_{Y_\beta} \\
X_\beta & \xleftarrow{\text{id}_{X_\beta}} & X_\beta & & & & \\
\downarrow d_{\beta'} & \swarrow h_\beta & \downarrow d_\beta & & & & \\
TY_{\beta'} & \xleftarrow{\mu_{Y_{\beta'}}} & T^2Y_{\beta'} & \xleftarrow{Td_{\beta'}} & TX_\beta & \xleftarrow{Ti_\beta} & TY_\beta
\end{array}$$

□

Above result simplifies in the case one restricts to an endomorphism: the respective basis representations are similar.

Corollary 4.6. *There exists a Kleisli isomorphism p with Kleisli inverse p^{-1} such that $f_{\alpha'\alpha'} = p^{-1} \cdot f_{\alpha\alpha} \cdot p$.*

Proof. In Proposition 4.2 let $\beta = \alpha$ and $\beta' = \alpha'$. One verifies that in the corresponding proof the definitions of the morphisms p^{-1} and q coincide. □

4.4 Bases as free algebras

In Corollary 4.1 it was shown that an algebra over a monad with a basis is isomorphic to a free algebra. Conversely, in Corollary 4.5 it was proven that a free algebra over a monad admits a basis. Intuitively, one may thus think that bases for an algebra coincide with free isomorphic algebras. In this section we make this idea precise on the level of categories.

Formally, given an algebra (X, h) over a monad \mathbb{T} , let $\text{Free}(X, h)$ denote the category defined as follows:

- objects are given by pairs (Y, φ) consisting of an isomorphism $\varphi : (TY, \mu_Y) \rightarrow (X, h)$; and

- a morphism $f : (Y_\alpha, \varphi_\alpha) \rightarrow (Y_\beta, \varphi_\beta)$ between objects consists of a morphism $f : Y_\alpha \rightarrow Y_\beta$ such that $\varphi_\alpha = \varphi_\beta \circ Tf$.

The next result shows that for a fixed algebra, the natural isomorphism underlying the free-algebra adjunction restricts to an equivalence between the category of bases defined in Definition 4.1, and the category of free isomorphic algebras given above.

Proposition 4.3. $\text{Bases}(X, h) \simeq \text{Free}(X, h)$

Proof. We define functors F and G between the respective categories as follows

$$\begin{aligned} F : \text{Bases}(X, h) &\longrightarrow \text{Free}(X, h) & F(Y, i, d) &= (Y, i^\sharp) & Ff &= f \\ G : \text{Free}(X, h) &\longrightarrow \text{Bases}(X, h) & G(Y, \varphi) &= (Y, \varphi \circ \eta_Y, \varphi^{-1}) & Gf &= f. \end{aligned}$$

The functor F is well-defined on objects since by Corollary 4.1 the morphism i^\sharp is an isomorphism with inverse d . Its well-definedness on morphisms is an immediate consequence of the constraint $i_\alpha = i_\beta \circ f$ for morphisms $f : (Y_\alpha, i_\alpha, d_\alpha) \rightarrow (Y_\beta, i_\beta, d_\beta)$ between bases,

$$(i_\alpha)^\sharp = h \circ Ti_\alpha = h \circ Ti_\beta \circ Tf = (i_\beta)^\sharp \circ Tf.$$

The functor G is well-defined on objects since $(\varphi \circ \eta_Y)^\sharp = \varphi$ and φ^{-1} are mutually inverse. Its well-definedness on morphisms $f : (Y_\alpha, \varphi_\alpha) \rightarrow (Y_\beta, \varphi_\beta)$ follows from the equality $\varphi_\beta \circ Tf = \varphi_\alpha$,

$$Tf \circ (\varphi_\alpha)^{-1} = (\varphi_\beta)^{-1} \circ \varphi_\beta \circ Tf \circ (\varphi_\alpha)^{-1} = (\varphi_\beta)^{-1} \circ \varphi_\alpha \circ (\varphi_\alpha)^{-1} = (\varphi_\beta)^{-1}$$

and the naturality of η ,

$$\varphi_\alpha \circ \eta_{Y_\alpha} = \varphi_\beta \circ Tf \circ \eta_{Y_\alpha} = \varphi_\beta \circ \eta_{Y_\beta} \circ f.$$

The functors are clearly mutually inverse on morphisms. For objects the statement follows from

$$\begin{aligned} F \circ G(Y, \varphi) &= (Y, (\varphi \circ \eta_Y)^\sharp) = (Y, \varphi) \\ G \circ F(Y, i, d) &= (Y, i^\sharp \circ \eta_Y, (i^\sharp)^{-1}) = (Y, i, d). \end{aligned}$$

□

4.5 Bases for bialgebras

It is well-known that a distributive law λ between a monad \mathbb{T} and an endofunctor F induces a monad \mathbb{T}_λ on the category of F -coalgebras such that the algebras over \mathbb{T}_λ coincide with the λ -bialgebras of Definition 2.5. This section is concerned with generators and bases for \mathbb{T}_λ -algebras, or equivalently, λ -bialgebras.

By definition, a generator for a λ -bialgebra (X, h, k) consists of a F -coalgebra (Y, k_Y) and morphisms $i : Y \rightarrow X$ and $d : X \rightarrow TY$, such that the three diagrams on the left below commute

$$\begin{array}{ccccc} Y & \xrightarrow{i} & X & & X & \xrightarrow{d} & TY & & TY & \xrightarrow{Ti} & TX & & TX & \xrightarrow{h} & X \\ k_Y \downarrow & & \downarrow k & & \downarrow k & & \lambda_Y \circ Tk_Y \downarrow & & \uparrow d & & \downarrow h & & \uparrow Ti & & \downarrow d \\ FY & \xrightarrow{Fi} & FX & & FX & \xrightarrow{Fd} & FTY & & X & \xrightarrow{\text{id}_X} & X & & TY & \xrightarrow{\text{id}_{TY}} & TY \end{array} \quad (6)$$

A basis for a λ -bialgebra is moreover given by a generator, such that in addition the diagram on the right above commutes.

It is easy to verify that by forgetting the F -coalgebra structure, every generator for a bialgebra in particular provides a generator for the underlying algebra of the bialgebra. By Proposition 3.1 it thus follows that there exists a λ -bialgebra homomorphism

$$i^\sharp = h \circ Ti : (TY, \mu_Y, (Fd \circ k \circ i)^\sharp) \longrightarrow (X, h, k).$$

The next result establishes that there exists a second equivalent free bialgebra with a different coalgebra structure.

Lemma 4.9. *Let (Y, k_Y, i, d) be a generator for (X, h, k) . Then $i^\sharp : TY \rightarrow X$ is a λ -bialgebra homomorphism $i^\sharp : (TY, \mu_Y, \lambda_Y \circ Tk_Y) \rightarrow (X, h, k)$.*

Proof. Clearly i^\sharp is a \mathbb{T} -algebra homomorphism. It is a F -coalgebra homomorphism since the diagram below commutes

$$\begin{array}{ccccc} TY & \xrightarrow{Ti} & TX & \xrightarrow{h} & X \\ Tk_Y \downarrow & & \downarrow Tk & & \downarrow k \\ TFY & \xrightarrow{TFi} & TFX & & \\ \lambda_Y \downarrow & & \downarrow \lambda_X & & \\ FTY & \xrightarrow{FTi} & FTX & \xrightarrow{Fh} & FX \end{array} .$$

□

If one moves from generator for bialgebras to bases for bialgebras, both coalgebra structures coincide.

Lemma 4.10. *Let (Y, k_Y, i, d) be a basis for (X, h, k) , then $\lambda_Y \circ Tk_Y = (Fd \circ k \circ i)^\sharp$.*

Proof. Using Lemma 4.1 we establish the commutativity of diagram below

$$\begin{array}{ccccccc} TY & \xrightarrow{Tk_Y} & TFY & \xrightarrow{\lambda_Y} & FTY & \xrightarrow{id_{FTY}} & FTY \\ \downarrow id_{TY} & & \downarrow TFi & & \downarrow FTi & & \downarrow id_{FTY} \\ TY & \xrightarrow{Ti} & TX & \xrightarrow{Tk} & TFX & \xrightarrow{TFd} & TFTY & \xrightarrow{\lambda_{TY}} & FT^2Y & \xrightarrow{F\mu_Y} & FTY \\ & & & & & & & & \downarrow FTd & \searrow Fd & \\ & & & & & & & & FTX & \xrightarrow{Fh} & FX \end{array} .$$

□

We close this section by observing that a basis for the underlying algebra of a bialgebra is sufficient for constructing a generator for the full bialgebra.

Lemma 4.11. *Let (X, h, k) be a λ -bialgebra and (Y, i, d) a basis for the \mathbb{T} -algebra (X, h) . Then $(TY, (Fd \circ k \circ i)^\sharp, i^\sharp, \eta_{TY} \circ d)$ is a generator for (X, h, k) .*

Proof. In the following we abbreviate $k_{TY} := (Fd \circ k \circ i)^\sharp$. By Proposition 3.1 the morphism i^\sharp is a F -coalgebra homomorphism $i^\sharp : (TY, k_{TY}) \rightarrow (X, k)$. This shows the commutativity of the diagram on the left of (6). By Proposition 4.1 the morphism d is a F -coalgebra homomorphism in the reverse direction. Together with the commutativity of the diagram on the left below

$$\begin{array}{ccc} TY & \xrightarrow{\eta_{TY}} & T^2Y \\ \downarrow k_{TY} & & \downarrow Tk_{TY} \\ & \nearrow \eta_{FTY} & TFTY \\ FTY & \xrightarrow{F\eta_{TY}} & FT^2Y \end{array} \quad \begin{array}{ccccc} T^2Y & \xrightarrow{T^2i} & T^2X & \xrightarrow{Th} & TX \\ \eta_{TY} \uparrow & & \eta_{TX} \uparrow & \searrow \mu_X & \downarrow h \\ TY & \xrightarrow{Ti} & TX & \xrightarrow{id_{TX}} & TX \\ d \uparrow & & \uparrow & & \downarrow h \\ X & \xrightarrow{id_X} & X & & X \end{array} .$$

this implies the commutativity of the second diagram to the left of (6). Similarly, the commutativity of third diagram to the left of (6) follows from the commutativity of the diagram on the right above.

□

5 Examples

In this section we give examples of generators and bases for algebras over the powerset-, downset-, distribution-, and neighbourhood monad.

5.1 Powerset monad

The powerset monad $\mathbb{P} = (\mathcal{P}, \mu, \eta)$ on the category of sets may easily be the most well-known monad. Its underlying set endofunctor \mathcal{P} assigns to a set its powerset and to a function the function that maps a subset to its direct image under the former. Multiplication and unit transformations are further given by

$$\eta_X(x) = \{x\} \quad \mu_X(U) = \bigcup_{A \in U} A.$$

The category of algebras for the powerset monad is famously isomorphic to the category of complete lattices and join-preserving functions. Indeed, since the existence of all joins is equivalent to the existence of all meets, one can define, given a \mathbb{P} -algebra (X, h) , a complete lattice (X, \leq) with $x \leq y \Leftrightarrow h(\{x, y\}) = y$ and supremum $\bigvee A := h(A)$. Conversely, given a complete lattice (X, \leq) with supremum \bigvee one defines a \mathbb{P} -algebra (X, h) by $h(A) := \bigvee A$.

Let F be the set endofunctor satisfying $FX = X^A \times 2$. Coalgebras for F are easily recognised as deterministic unpointed automata over the alphabet A , and coalgebras for the composition $F\mathcal{P}$ coincide with unpointed non-deterministic automata over the same alphabet. The output set 2 can be equipped with a disjunctive \mathbb{P} -algebra structure, such that bialgebras for the induced canonical distributive law between \mathbb{P} and F consist of deterministic automata with a complete lattice as state space and supremum-preserving transition and output functions [18]. For instance, every deterministic automaton derived from a non-deterministic automaton via the classical subset construction is of such a form.

The following observation relates generators for algebras over \mathbb{P} with the induced complete lattice structure of the former.

Lemma 5.1. *(Y, i, d) is a generator for a \mathbb{P} -algebra (X, h) iff*

$$x = \bigvee_{y \in d(x)} i(y)$$

for all $x \in X$.

Proof. Follows immediately from the equality

$$h \circ \mathcal{P}i \circ d(x) = h(\{i(y) \mid y \in d(x)\}) = \bigvee_{y \in d(x)} i(y).$$

□

Recall that a non-zero element $x \in X$ of a lattice $L = (X, \leq)$ is called join-irreducible, if $x = y \vee z$ implies $x = y$ or $x = z$. It is well-known that in a finite lattice, or more generally in a lattice satisfying the descending chain condition, any element is the join of the join-irreducible elements below it. In other words, by Lemma 5.1, if $i : \mathcal{J}(L) \rightarrow L$ is the subset embedding of join-irreducibles and $d : L \rightarrow \mathcal{P}(\mathcal{J}(L))$ satisfies

$$d(x) = \{a \in \mathcal{J}(L) \mid a \leq x\},$$

then $(\mathcal{J}(L), i, d)$ is a generator for the \mathbb{P} -algebra L .

5.2 Downset monad

In the previous section we have seen that the category of algebras for the powerset monad is equivalent to the category of complete lattices and join-preserving functions. It is probably slightly less-known that there exists a second monad with the same category of algebras: the downset monad $\mathbb{P}_\downarrow = (\mathcal{P}_\downarrow, \mu, \eta)$ on the category of posets.

For a subset Y of a poset let Y_\downarrow be its so-called downward closure, that is, the set of those poset elements for which there exists at least one element in Y above. A subset is called downset or downclosed, if it coincides with its downward closure. The endofunctor \mathcal{P}_\downarrow underlying the downset monad assigns to a poset the inclusion-ordered poset of its downclosed subsets, and to a monotone function the monotone function mapping a downclosed subset to the downclosure of its direct image. The natural transformations η and μ are further given by

$$\eta_P(x) = \{x\}_\downarrow \quad \mu_P(U) = \bigcup_{A \in U} A.$$

Given an algebra (P, h) over \mathbb{P}_\downarrow , one verifies that the poset P is a complete lattice with supremum $\bigvee A := h(A_\downarrow)$. Conversely, given a complete lattice P with supremum \bigvee , one defines an algebra (P, h) over \mathbb{P}_\downarrow by $h(A) := \bigvee A$. The following observation relates generators for algebras over \mathbb{P}_\downarrow with the induced complete lattice structure of the latter.

Lemma 5.2. *(Y, i, d) is a generator for a \mathbb{P}_\downarrow -algebra (P, h) iff*

$$x = \bigvee_{y \in d(x)} i(y)$$

for all $x \in X$.

Proof. Follows immediately from the equality

$$h \circ \mathcal{P}_\downarrow i \circ d(x) = h(\{i(y) \mid y \in d(x)\}_\downarrow) = \bigvee_{y \in d(x)} i(y).$$

□

It is due to Birkhoff's Representation Theorem that if L is a finite (hence complete) distributive lattice, the monotone function $d : L \rightarrow \mathcal{P}_\downarrow(\mathcal{J}(L))$ assigning to an element the downward closed set of join-irreducibles below it, is an isomorphism with an inverse satisfying $A \mapsto \bigvee A$. In other words, in light of the previous result, if $i : \mathcal{J}(L) \rightarrow L$ is the subset embedding of join-irreducibles, then $(\mathcal{J}(L), i, d)$ constitutes a basis for the \mathbb{P}_\downarrow -algebra L .

5.3 Distribution monad

The distribution monad $\mathbb{D} = (\mathcal{D}, \mu, \eta)$ on the category of sets is given as follows. The underlying set endofunctor \mathcal{D} assigns to a set X its set of distributions with finite support,

$$\mathcal{D}X = \{p : X \rightarrow [0, 1] \mid \text{supp}(p) \text{ finite, and } \sum_{x \in X} p(x) = 1\}$$

and to a function $f : X \rightarrow Y$ the direct image $\mathcal{D}f : \mathcal{D}X \rightarrow \mathcal{D}Y$ satisfying the equality

$$\mathcal{D}(f)(p)(y) = \sum_{x \in f^{-1}(y)} p(x).$$

The natural transformations η and μ are further given by

$$\eta_X(x)(y) = [x = y] \quad \mu_X(\Phi)(x) = \sum_{p \in \mathcal{D}X} p(x)\Phi(p).$$

It is well-known that the category of algebras for the distribution monad is isomorphic to the category of convex sets and affine functions [26]. Indeed, any \mathbb{D} -algebra (X, h) can be turned into a unique convex set with finite sums defined by $\sum_i r_i x_i := h(p)$ for $p(x) = r_i$ if $x = x_i$, and $p(x) = 0$ otherwise.

Let F be the set endofunctor satisfying $FX = X^A \times [0, 1]$. Coalgebras for the composed endofunctor $F\mathcal{D}$ are known as unpointed Rabin probabilistic automata over the alphabet A [27, 19]. The unit interval can be equipped with a \mathbb{D} -algebra structure h which satisfies $h(p) = \sum_{x \in [0, 1]} xp(x)$

and induces a canonical distributive law between \mathbb{D} and F [18]. The respective bialgebras consist of unpointed Moore automata with input A and output $[0, 1]$, a convex set as state space, and affine transition and output functions. For instance, every Moore automaton derived from a probabilistic automaton by assigning to the state space of the latter all its distributions with finite support constitutes such a bialgebra.

The next observation relates generators of algebras over \mathbb{D} with the induced convex set structure on the latter. For simplicity we assume that in the following statement the function $i : Y \rightarrow X$ is an injection.

Lemma 5.3. *(Y, i, d) is a generator for a \mathbb{D} -algebra (X, h) iff*

$$x = \sum_{y \in Y} d(x)(y)i(y)$$

for all $x \in X$.

Proof. For $x \in X$ let $p_x \in \mathcal{D}X$ be the distribution satisfying the equality $p_x(\bar{x}) = \sum_{y \in i^{-1}(\bar{x})} d(x)(y)$. Since i is injective we find $p_x(\bar{x}) = d(x)(y)$, if $\bar{x} = i(y)$, and $p_x(\bar{x}) = 0$ otherwise. Thus we can deduce

$$h \circ \mathcal{D}i \circ d(x) = h(p_x) = \sum_{y \in Y} d(x)(y)i(y).$$

□

5.4 Neighbourhood monad

It is well-known that the contravariant setfunctor assigning to a set its powerset, and to a function the function that precomposes a characteristic function with the former, is dually self-adjoint. The monad $\mathbb{H} = (N, \mu, \eta)$ induced by the adjunction is known under the name neighbourhood monad, since its coalgebras are related to neighbourhood frames in modal logic [28]. Its underlying set endofunctor N is given by

$$NX = 2^{2^X} \quad Nf(\Phi)(\varphi) = \Phi(\varphi \circ f),$$

and its unit η and multiplication μ satisfy

$$\eta_X(x)(\varphi) = \varphi(x) \quad \mu_X(\Phi)(\varphi) = \Phi(\eta_{2^X}(\varphi)).$$

Recall that a non-zero element of a lattice is called atomic, if there exists no non-zero element below it, and a lattice is called atomic, if each element can be written as join of atoms below it. It is well-known that (i) the category of algebras over \mathbb{H} is equivalent to the category of complete atomic Boolean algebras; and (ii) the category of complete atomic Boolean algebras is contravariant equivalent to the category of sets [29].

In more detail [30], the equivalence (i) assigns to a \mathbb{H} -algebra (X, h) the complete atomic Boolean algebra on X with pointwise induced operations

$$\begin{aligned} 0 &= h(\emptyset) & 1 &= h(2^X) & \neg x &= h(\sim \eta_X(x)) \\ \bigvee A &= h\left(\bigcup_{x \in A} \eta_X(x)\right) & \bigwedge A &= h\left(\bigcap_{x \in A} \eta_X(x)\right), \end{aligned} \tag{7}$$

while the equivalence (ii) assigns to a complete atomic Boolean algebra B its set of atoms $At(B)$, and to a set X the complete atomic Boolean powerset algebra $\mathcal{P}X$. The mutually invertibility of the assignments in the latter equivalence is witnessed by the Boolean algebra isomorphism between B and $\mathcal{P}(At(B))$ that assigns to an element the set of atoms below it. For K the Eilenberg-Moore comparison functor induced by the monadic self-dual powerset adjunction, and J the equivalence in (i), one may recover the representation in (ii) as the composition of J with K . [29].

As before, let F be the set endofunctor satisfying $FX = X^A \times 2$. One verifies that coalgebras for the composed endofunctor FN can be recognised as unpointed alternating automata [30]. The set 2 can be equipped with a \mathbb{H} -algebra structure h which satisfies $h(\varphi) = \varphi(\text{id}_2)$ and induces a canonical

distributive law between \mathbb{H} and F [18]. The corresponding bialgebras consist of deterministic automata with a complete atomic Boolean algebra as state space and join-preserving transition and output Boolean algebra homomorphisms. For instance, every deterministic automaton derived from an alternating automaton via a double subset construction is of such a form.

The next observation relates generators of a \mathbb{H} -algebras with the induced complete atomic Boolean algebra structure on the latter.

Lemma 5.4. *(Y, i, d) is a generator for a \mathbb{H} -algebra (X, h) iff*

$$x = \bigvee_{A \in d(x)} \left(\bigwedge_{y \in A} i(y) \wedge \bigwedge_{y \notin A} \neg i(y) \right)$$

for all $x \in X$.

Proof. One verifies that after identifying a subset with its characteristic function, any $\Phi \in NY$ satisfies

$$\Phi(\psi) = \bigvee_{A \in \Phi} \left(\bigwedge_{y \in A} \psi(y) \wedge \bigwedge_{y \notin A} \neg \psi(y) \right).$$

In particular we can deduce the following equality,

$$\begin{aligned} x &= h \circ Ni \circ d(x) = h(\lambda\varphi.d(x)(\varphi \circ i)) \\ &= h\left(\bigcup_{A \in d(x)} \left(\bigcap_{y \in A} \eta_X(i(y)) \cap \bigcap_{y \notin A} \sim \eta_X(i(y)) \right) \right). \end{aligned}$$

The statement follows from the latter by (7). □

6 Related work

One of the main motivations for the present paper has been our broad interest in active learning algorithms for state-based models [31], in particular automata for NetKAT [32], a formal system for the verification of networks based on Kleene Algebra with Tests [33]. One of the main challenges in learning non-deterministic models such as NetKAT automata is the common lack of a unique minimal acceptor for a given language [24]. The problem has been independently approached for different variants of non-determinism, often with the common idea of finding a subclass admitting a unique representative [10, 11]. A more general and unifying perspective has been given in [12] by van Heerdt, see also [21, 22].

One of the central notions in the work of van Heerdt is the concept of a scoop, originally introduced by Arbib and Manes [13]. In the present paper the notion coincides with what we call a generator in Definition 3.1. Scoops have primarily been used as a tool for constructing minimal realisations of automata, similarly to Proposition 3.1. Strengthening the definition of Arbib and Manes to the notion of a basis in Definition 4.1 allows us to further extend such automata-theoretical results, e.g. Corollary 4.4, but also uncovers ramifications with universal algebra, leading for instance to a representation theory of algebra homomorphisms in the same framework.

A generalisation of the notion of a basis to algebras of arbitrary monads has been approached before. For instance, in [25] Jacobs defines a basis for an algebra as a coalgebra for the comonad on the category of algebras induced by the free algebra adjunction. One can show that a basis in the sense of Definition 4.1 always induces a basis in the sense of [25]. Conversely, given certain assumptions about the existence and preservation of equaliser, it is possible to recover a basis in the sense of Definition 4.1 from a basis in the sense of [25]. Starting with a basis in the sense of Definition 4.1, the composition of both translations yields a basis that is not less compact than the basis one began with; in certain cases they coincide. As equaliser do not necessarily exist and are not necessarily preserved, our approach carries additional data and thus can be seen as finer.

7 Discussion and future work

We have presented a notion of a basis for an algebra over a monad on an arbitrary category that subsumes the familiar notion for algebraic theories. We have covered questions about the existence and uniqueness of bases, and established a representation theory for homomorphisms between algebras over a monad in the spirit of linear algebra by substituting Kleisli morphisms for matrices. Building on foundations in the work of Arbib and Manes [13], we further have established that a basis for the underlying algebra of a bialgebra yields an isomorphic bialgebra with free state space. Moreover, we have established an equivalence between the category of bases for an algebra and the category of its isomorphic free algebras, and looked into bases for bialgebras. Finally we gave characterisations of bases for algebras over the powerset, downset, distribution, and neighbourhood monad.

For the future we are particularly interested in using the present work for a unified view on the theory of residual finite state automata [24] (RFSA) and variations of it, for instance the theories of residual probabilistic automata [10] and residual alternating automata [11]. RFSA are non-deterministic automata that share with deterministic automata two important properties: for any regular language they admit a unique minimal acceptor, and the language of each state is a residual of the language of its initial state. In Example 3.2 we have demonstrated that the so-called canonical RFSA can be recovered as the bialgebra with free state space induced by a generator of join-irreducibles for the underlying algebra of a particular bialgebra. We believe we can uncover similar correspondences for other variations of non-determinism. Similar ideas have already been served as motivation in the work of Arbib and Manes [13] and have recently come up again in [23]. We are also interested in insights into the formulation of active learning algorithms along the lines of [31] for different classes of residual automata, as sketched in the related work section.

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Appendix

A Bases as coalgebras

This section is concerned with an alternative approach to the generalisation of bases. More specifically, we are interested in the work of Jacobs [25], where a basis for an algebra over a monad is defined as a coalgebra for the comonad on the category of Eilenberg-Moore algebras induced by the free algebra adjunction. Explicitly, a basis for a \mathbb{T} -algebra (X, h) in the former sense consists of a T -coalgebra (X, k) such that the following three diagrams commute:

$$\begin{array}{ccc}
 TX & \xrightarrow{Tk} & T^2X \\
 \downarrow h & & \downarrow \mu_X \\
 X & \xrightarrow{k} & TX
 \end{array}
 \quad
 \begin{array}{ccc}
 X & \xrightarrow{k} & TX \\
 \searrow \text{id}_X & & \downarrow h \\
 & & X
 \end{array}
 \quad
 \begin{array}{ccc}
 X & \xrightarrow{k} & TX \\
 \downarrow k & & \downarrow T\eta_X \\
 TX & \xrightarrow{Tk} & T^2X
 \end{array}
 \quad (8)$$

The next result shows that a basis in the sense of Definition 4.1 induces a basis in the sense of [25].

Lemma A.1. *Let (Y, i, d) be a basis for a \mathbb{T} -algebra (X, h) . Then (8) commutes for $k := Ti \circ d$.*

Proof. The commutativity of the diagram on the left of (8) follows directly from Corollary 4.2. The diagram in the middle of (8) commutes by the definition of a generator. The commutativity of the diagram on the right of (8) is a consequence of Lemma 4.1,

$$\begin{array}{ccccc}
 X & \xrightarrow{d} & TY & \xrightarrow{Ti} & TX \\
 \downarrow d & \nearrow \text{id}_{TY} & & & \downarrow T\eta_X \\
 TY & & & & \\
 \downarrow Ti & \searrow T\eta_Y & & & \\
 TX & \xrightarrow{Td} & T^2Y & \xrightarrow{T^2i} & T^2X
 \end{array}$$

□

Above construction easily extends to morphisms between bases. Indeed, assume there exists a morphism $f : (Y_\alpha, i_\alpha, d_\alpha) \rightarrow (Y_\beta, i_\beta, d_\beta)$ of bases for a \mathbb{T} -algebra (X, h) . Then the identity morphism is a T -coalgebra homomorphism $\text{id}_X : (X, Ti_\alpha \circ d_\alpha) \rightarrow (X, Ti_\beta \circ d_\beta)$ as the following diagram shows

$$\begin{array}{ccccc}
 X & \xrightarrow{d_\alpha} & TY_\alpha & \xrightarrow{Ti_\alpha} & TX \\
 \downarrow \text{id}_X & & \downarrow Tf & & \downarrow \text{id}_{TX} \\
 X & \xrightarrow{d_\beta} & TY_\beta & \xrightarrow{Ti_\beta} & TX
 \end{array}$$

Conversely, assume (X, k) is a T -coalgebra structure satisfying (8) and $i_k : Y \rightarrow X$ an equaliser of k and η_X . If the underlying category is the category of sets and functions, and Y non-empty, one can show that the equaliser is preserved under T , that is, Ti_k is an equaliser of Tk and $T\eta_X$ [25]. Since it holds $Tk \circ k = T\eta_X \circ k$ by (8), there thus exists a unique morphism $d_k : X \rightarrow TY$, which can be shown to be the inverse of $h \circ Ti_k$ [25]. In other words, $G(X, k) := (Y, i_k, d_k)$ is a basis for (X, h) in the sense of Definition 4.1. If we further let $F(Y, i, d) := Ti \circ d$ for an arbitrary basis of (X, h) , then the result below can be used to show that there exists a unique morphism $(Y, i, d) \rightarrow GF(Y, i, d)$ of bases. This suggests an adjoint relation between the two different notions.

Lemma A.2. *Let (Y, i, d) be a basis for a \mathbb{T} -algebra (X, h) . Then $(Ti \circ d) \circ i = \eta_X \circ i$.*

Proof. The statement follows from Lemma 4.1 and the naturality of η ,

$$\begin{array}{ccc} Y & \xrightarrow{i} & X \\ \downarrow i & \searrow \eta_Y & \downarrow \eta_X \\ X & \xrightarrow{d} TY & \xrightarrow{Ti} TX \end{array}$$

□