Canonical automata via distributive law homomorphisms

Stefan Zetzsche [1]   Gerco von Heerdt [1]   Alexandra Silva [1]
Matteo Sammartino [1,2]

[1]University College London

[2]Royal Holloway University of London

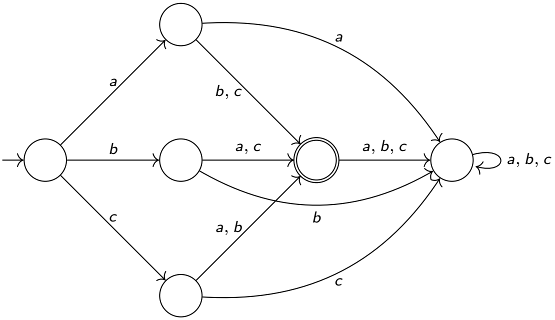
May 10, 2021

[0]https://arxiv.org/abs/2104.13421

Introduction

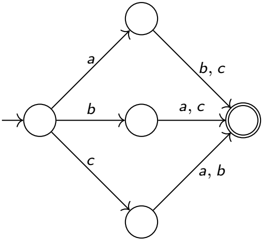
Minimal DFA

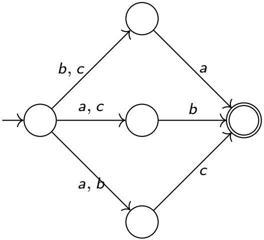
Up to isomorphism, the unique size-minimal DFA accepting $\mathcal{L} = \{ab, ac, ba, bc, ca, cb\} \subseteq \{a, b, c\}^*$:



Minimal NFA

Two non-isomorphic[1] size-minimal NFA accepting
$\mathcal{L} = \{ab, ac, ba, bc, ca, cb\} \subseteq \{a, b, c\}^*$:



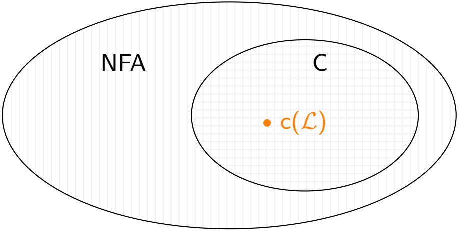


Is there a canonical NFA for $\mathcal{L}$?

[1] Arnold, Dicky, and Nivat 1992.

Minimal NFA

Is there a subclass $C \subseteq$ NFA, such that any regular language $\mathcal{L}$ admits a canonical acceptor $c(\mathcal{L}) \in C$ size-minimal in C?



Example: The canonical RFSA

A NFA accepting $\mathcal{L} \subseteq A^*$ is RFSA, if every state accepts a residual $u^{-1}\mathcal{L} = \{v \in A^* \mid uv \in \mathcal{L}\}$ for some $u \in A^*$.

The canonical RFSA for a regular language $\mathcal{L} \subseteq A^*$ is the X_0-pointed NFA $\langle \varepsilon, \delta \rangle : X \to 2 \times \mathcal{P}(X)^A$ given by:

- $X = \{U \subseteq A^* \mid U \text{ prime residual of } \mathcal{L}\}$;
- $X_0 = \{U \in X \mid U \subseteq \mathcal{L}\}$;
- $\varepsilon(U) = [\varepsilon \in U]$;
- $\delta_a(U) = \{V \in X \mid V \subseteq a^{-1}U\}$.

Theorem ([2])

The canonical RFSA for $\mathcal{L}$ is size-minimal among RFSA for $\mathcal{L}$.

[2] Denis, Lemay, and Terlutte 2002.

Example: The canonical RFSA

How does one come up with this definition? Why does it work?

NFA → DFA

The classical powerset construction converts a NFA into an equivalent DFA.

$$\langle \varepsilon, \delta \rangle : Y \to 2 \times \mathcal{P}(Y)^A$$

$$\downarrow$$

$$\langle \varepsilon^\sharp, \delta^\sharp \rangle^3 : \mathcal{P}(Y) \to 2 \times \mathcal{P}(Y)^A$$

[3] $\varepsilon^\sharp(U) = \bigvee_{u \in U} \varepsilon(u), \quad \delta_a^\sharp(U) = \bigcup_{u \in U} \delta_a(u)$

NFA $\to$ DFA (in CSL)

$$\varepsilon^\sharp(U_1 \cup U_2) = \varepsilon^\sharp(U_1) \vee \varepsilon^\sharp(U_2)$$
$$\delta_a^\sharp(U_1 \cup U_2) = \delta_a^\sharp(U_1) \cup \delta_a^\sharp(U_2)$$

$\langle \varepsilon^\sharp, \delta^\sharp \rangle$ is a DFA in the category of complete semilattices (CSL).

DFA (in CSL) $\to$ NFA

Consider the reverse to the powerset construction.

$$\begin{array}{c} \langle D, E \rangle : X \to X^A \times 2 \\ \big\downarrow{}^{4} \\ \langle \delta, \varepsilon \rangle : Y \to \mathcal{P}(Y)^A \times 2 \end{array} \qquad 2, X \in \text{CSL}$$

Possible? Maybe, choose Y as a generator for X? Can we find a size-minimal generator Y?

[4] Constraint: $\langle D, E \rangle \sim \langle \delta^\sharp, \varepsilon^\sharp \rangle$

T-DFA $\to$ T-NFA

Generalises[5] to other algebraic theories T:

$$\begin{array}{c} X \to X^A \times B \\ \downarrow \\ Y \to T(Y)^A \times B \end{array} \qquad B, X \in \mathrm{Alg}(T)$$

Related to the construction of canonical (minimal) automata:

- canonical RFSA[6] (T=CSL, B=2)
- canonical nominal RFSA[7] (T=Nominal CSL, B=2)
- minimal xor automaton[8] (T=$\mathbb{Z}_2$-VSP, B=2)

[5] Zetzsche, Silva, and Sammartino 2020.

[6] Denis, Lemay, and Terlutte 2002.

[7] Moerman and Sammartino 2019.

[8] Vuillemin and Gama 2010.

Example: The átomaton

Previous work[9] is not general enough to capture e.g. the átomaton[10], which intertwines CABA and CSL.

The átomaton for a regular language $\mathcal{L} \subseteq A^*$ is the X_0-pointed NFA $\langle \varepsilon, \delta \rangle : X \to 2 \times \mathcal{P}(X)^A$ given by:

- $X = \{U \subseteq A^* \mid U \text{ atom of } \mathcal{L}\}$;
- $X_0 = \{U \in X \mid U \subseteq \mathcal{L}\}$;
- $\varepsilon(U) = [\varepsilon \in U]$;
- $\delta_a(U) = \{V \in X \mid V \subseteq a^{-1}U\}$.

[9] Zetzsche, Silva, and Sammartino 2020.

[10] Brzozowski and Tamm 2014.

S-DFA $\to$ T-NFA

Need a situation parametric in two theories S, T:

$$\begin{array}{c} X \to X^A \times B \\ \big\downarrow \\ Y \to T(Y)^A \times B \end{array} \qquad B, X \in \mathrm{Alg}(S)$$

Rough idea:

- átomaton (S = CABA, T = CSL, B = 2)
- distromaton[11] (S = CDL, T = CSL, B = 2)
- ...

[11] Myers et al. 2015.

Contributions

- Categorical framework for the derivation of canonical automata
- Strictly improve expressivity of previous work
- Cover categories different from set, e.g. nominal sets
- Discover a new canonical acceptor by relating mod-2 vector spaces with CABAs
- Present sufficient conditions for the existence of minimal acceptors
- Subsume and establish new minimality results

Plan

Preliminaries

- Monads, distributive laws, bialgebras

Previous work

- Generators for (bi)algebras
- Example: The canonical RFSA

Current work

- (Deriving) distributive law homomorphisms
- Example: The átomaton
- The minimal xor-CABA automaton
- Minimality

Preliminaries

Overview

T-algebra	$TX \to X \in \mathrm{Alg}(T)$
free T-algebra	$T^2Y \to TY \in \mathrm{Alg}(T)$
DFA	$X \to FX \in \mathrm{Coalg}(F)$
T-DFA	$TX \to X \to FX \in \mathrm{Bialg}(\lambda)$
T-NFA	$T^2Y \to TY \to FTY \in \mathrm{Bialg}(\lambda)$

Monads

The examples below are the most relevant ones for us.

- The powerset monad with $\mathcal{P}X = 2^X$ and $\mathsf{Alg}(\mathcal{P}) = \mathsf{CSL}$;
- The free mod-2 vector space monad with $\mathcal{X}X = 2^X$ and $\mathsf{Alg}(\mathcal{X}) = \mathbb{Z}_2\text{-}\mathsf{VSP}$;
- The neighbourhood monad with $\mathcal{H}X = 2^{2^X}$ and $\mathsf{Alg}(\mathcal{H}) = \mathsf{CABA}$;
- The monotone neighbourhood monad with $\mathcal{A}X = (2, \leq)^{(2^X, \subseteq)}$ and $\mathsf{Alg}(\mathcal{A}) = \mathsf{CDL}$.

Distributive laws

A distributive law between a monad $\langle T, \eta, \mu \rangle$ on C and an endofunctor $F : \mathsf{C} \to \mathsf{C}$ is a natural transformation

$$\lambda : TF \Rightarrow FT$$

satisfying the laws

$$\lambda \circ \eta_F = F\eta \qquad \lambda\mu_F = F\mu \circ \lambda_T \circ T\lambda.$$

[11] For example, if F satisfies $FX = B \times X^A$ and $\langle B, h \rangle$ is a T-algebra, the family

$$(\lambda^h)_X : T(B \times X^A) \overset{\langle T\pi_1, T\pi_2 \rangle}{\to} TB \times T(X^A) \overset{h \times \mathsf{st}}{\to} B \times (TX)^A$$

induces a distributive law between T and F.

Bialgebras

A λ-bialgebra is a tuple $\langle X, h, k \rangle$ consisting of an object X and morphisms

$$TX \xrightarrow{h} X \in \mathsf{Alg}(T), \qquad X \xrightarrow{k} FX \in \mathsf{Coalg}(F)$$

satisfying

$$\begin{array}{ccccc} TX & & \xrightarrow{\quad h \quad} & & X \\ {\scriptstyle Tk}\downarrow & & & & \downarrow{\scriptstyle k} \\ TFX & \xrightarrow{\lambda_X} & FTX & \xrightarrow{Fh} & FX \end{array} .$$

Previous work

Generators

A generator[12] for a T-algebra $\langle X, h \rangle$ is a tuple $\langle Y, i, d \rangle$ consisting of an object Y and a pair of morphisms

$$TY \underset{d}{\overset{i^\sharp}{\rightleftarrows}} X \qquad \text{satisfying} \qquad i^\sharp \circ d = \mathrm{id}_X,$$

where $i^\sharp := h \circ Ti : TY \to X$ is the unique extension of $i : Y \to X$ to a T-algebra homomorphism[13].

If in addition $d \circ i^\sharp = \mathrm{id}_{TY}$, we speak of a basis[14].

[12] Arbib and Manes 1975.

[13] For instance, every T-algebra $\langle X, h \rangle$ is generated by $\langle X, \mathrm{id}_X, \eta_X \rangle$.

[14] Zetzsche, Silva, and Sammartino 2020.

Generators

$\langle Y, i, d\rangle$ is a generator for an algebra $\langle X, h\rangle$ over the powerset monad iff for all $x \in X$

$$x = \bigvee_{y \in d(x)}^{h} i(y).$$

$\langle Y, i, d\rangle$ is a generator for an algebra $\langle X, h\rangle$ over the free mod-2 vector-space monad iff for all $x \in X$

$$x = \bigoplus_{y \in Y}^{h} d(x)(y) \cdot^{h} i(y).$$

Generators

Let $\langle X, h, k\rangle$ be a λ-bialgebra and $\langle Y, i, d\rangle$ a generator for the T-algebra $\langle X, h\rangle$.

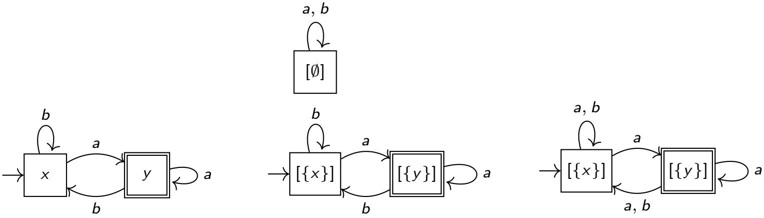
Lemma ([15])

The morphism $h \circ Ti : TY \to X$ is a λ-bialgebra homomorphism

$$h \circ Ti : \langle TY, \mu_Y, (Fd \circ k \circ i)^\sharp\rangle \to \langle X, h, k\rangle.$$

[15] Zetzsche, Silva, and Sammartino 2020.

Example: The canonical RFSA

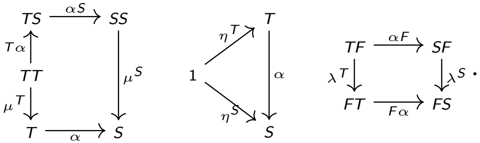


(a) (b) (c)

Figure:
(a) The minimal DFA for $\mathcal{L} = (a+b)^* a$;
(b) The minimal CSL-structured DFA $\langle X, h, k \rangle$ for $\mathcal{L}$;
(c) The canonical RFSA $\langle J(\langle X, h \rangle), Fd \circ k \circ i \rangle$ for $\mathcal{L}$.

Current work

A distributive law homomorphism[16] $\alpha : \lambda^S \to \lambda^T$ between $\lambda^S : SF \Rightarrow FS$ and $\lambda^T : TF \Rightarrow FT$ consists of a natural transformation $\alpha : T \Rightarrow S$ satisfying:



Lemma ([17])

Let $\alpha : \lambda^S \to \lambda^T$ be a distributive law homomorphism. Then $\alpha\langle X, h, k\rangle := \langle X, h \circ \alpha_X, k\rangle$ and $\alpha(f) := f$ defines a functor $\alpha : \mathrm{Bialg}(\lambda^S) \to \mathrm{Bialg}(\lambda^T)$.

[16] Watanabe 2002; Power and Watanabe 2002.

[17] Klin and Nachyla 2015; Bonsangue et al. 2013.

Distributive law homomorphisms

The following can be seen as roadmap or soundness argument to our approach.

Corollary

Let $\alpha : \lambda^S \to \lambda^T$ *be a homomorphism between distributive laws and* $\langle X, h, k\rangle$ *a* λ^S*-bialgebra. If* $\langle Y, i, d\rangle$ *is a generator for the* T*-algebra* $\langle X, h \circ \alpha_X\rangle$*, then*

$$(h \circ \alpha_X) \circ Ti : \langle TY, \mu_Y, (Fd \circ k \circ i)^\sharp\rangle \to \langle X, h \circ \alpha_X, k\rangle$$

is a λ^T*-bialgebra homomorphism.*

Deriving distributive law homomorphisms

If the distributive laws are induced by algebras $h^S : SB \to B$ and $h^T : TB \to B$, respectively, then deriving a homomorphism simplifies.

Lemma

Let $\alpha : T \to S$ be a natural transformation satisfying $h^S \circ \alpha_B = h^T$, then $\lambda^{h^S} \circ \alpha F = F\alpha \circ \lambda^{h^T}$.

Deriving distributive law homomorphisms

For the neighbourhood monad $\mathcal{H}$, there exists a parametrised family of canonical homomorphisms:

Corollary

Any algebra $h^T : T2 \to 2$ over a set monad T induces a homomorphism $\alpha^{h^T} : \lambda^{h^{\mathcal{H}}} \to \lambda^{h^T}$ between distributive laws by $\alpha_X^{h^T} := (h^T)^{2^X} \circ \mathsf{st} \circ T(\eta_X^{\mathcal{H}}) : TX \to \mathcal{H}X$.

Can be lifted to strong monads and arbitrary output objects on general categories.

Example: The átomaton

The átomaton can be recovered by relating $\mathcal{H}$ and $\mathcal{P}$.

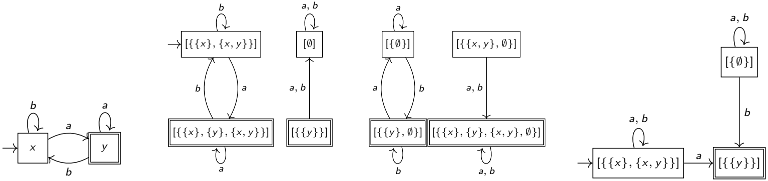
Corollary

Let $\alpha_X : \mathcal{P}X \to \mathcal{H}X$ *satisfy* $\alpha_X(\varphi)(\psi) = \bigvee_{x \in X} \varphi(x) \wedge \psi(x)$*, then* α *constitutes a distributive law homomorphism* $\alpha : \lambda^{h^{\mathcal{H}}} \to \lambda^{h^{\mathcal{P}}}$*.*

Lemma

Let $\alpha_X : \mathcal{P}X \to \mathcal{H}X$ *satisfy* $\alpha_X(\varphi)(\psi) = \bigvee_{x \in X} \varphi(x) \wedge \psi(x)$*. If* $B = \langle X, h \rangle$ *is a* $\mathcal{H}$*-algebra, then* $\langle \mathrm{At}(B), i, d \rangle$ *with* $i(a) = a$ *and* $d(x) = \{a \in \mathrm{At}(B) \mid a \leq x\}$ *is a basis for the* $\mathcal{P}$*-algebra* $\langle X, h \circ \alpha_X \rangle$*.*

Example: The átomaton



(a) (b) (c)

Figure:
(a) The minimal DFA for $\mathcal{L} = (a+b)^*a$;
(b) The minimal CABA-structured DFA $\langle X, h, k\rangle$ for $\mathcal{L}$;
(c) The átomaton $\langle \mathrm{At}(\langle X, h\rangle), Fd \circ k \circ i\rangle$ for $\mathcal{L}$.

The minimal xor-CABA automaton

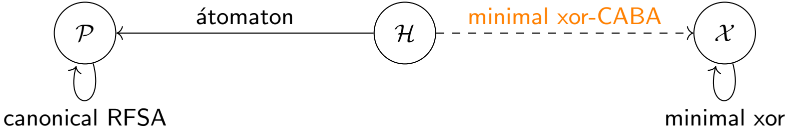
Corollary

Let $\alpha_X : \mathcal{X}X \to \mathcal{H}X$ *satisfy* $\alpha_X(\varphi)(\psi) = \bigoplus_{x \in X} \varphi(x) \cdot \psi(x)$*, then* α *constitutes a distributive law homomorphism* $\alpha : \lambda^{h^{\mathcal{H}}} \to \lambda^{h^{\mathcal{X}}}$*.*

Allows the definition of a new canonical acceptor – the minimal xor-CABA automaton.

The minimal xor-CABA automaton

"The minimal xor-CABA automaton is to the minimal xor automaton what the átomaton is to the canonical RFSA":

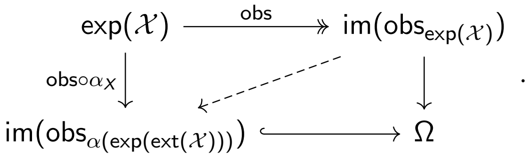


Minimality

In the presence of a distributive law there exist two possible semantics for automata with side-effects in T.

Definition

Let $\alpha : \lambda^S \to \lambda^T$ be a distributive law homomorphism. We say $\mathcal{X} \in \mathsf{Coalg}(FT)$ is α-closed if the unique diagonal below is an isomorphism:



---

[17] $\mathsf{ext} : \mathsf{Coalg}(FT) \to \mathsf{Coalg}(FS)$

[17] $\exp : \mathsf{Coalg}(FU) \to \mathsf{Bialg}(\lambda^U)$ for $U \in \{S, T\}$

[17] $\alpha : \mathsf{Bialg}(\lambda^S) \to \mathsf{Bialg}(\lambda^T)$

Minimality

Our main result provides sufficient conditions for the existence of a canonical minimal acceptor.

Theorem

Given a minimal λ^S-bialgebra $\mathbb{M}$ accepting $\mathcal{L}$ such that $\alpha(\mathbb{M})$ admits a size-minimal generator, there exists a α-closed T-succinct automaton $\mathcal{X}$ accepting $\mathcal{L}$ such that:

- *for any α-closed T-succinct automaton $\mathcal{Y}$ accepting $\mathcal{L}$ we have that* $\mathrm{im}(\mathrm{obs}_{\mathrm{exp}(\mathcal{X})}) \subseteq \mathrm{im}(\mathrm{obs}_{\mathrm{exp}(\mathcal{Y})})$;
- *if* $\mathrm{im}(\mathrm{obs}_{\mathrm{exp}(\mathcal{X})}) = \mathrm{im}(\mathrm{obs}_{\mathrm{exp}(\mathcal{Y})})$, *then* $|\mathcal{X}| \leq |\mathcal{Y}|$.

Minimality

For the cases below α is given by the identity, thus rendering closedness trivial.

Corollary

- *The canonical RFSA for $\mathcal{L}$ is size-minimal among NFA $\mathbb{Y}$ accepting $\mathcal{L}$ with* $\overline{\mathrm{im}(\mathrm{obs}^{\dagger}_{\mathbb{Y}})}^{\mathsf{CSL}} \subseteq \overline{\mathrm{Der}(\mathcal{L})}^{\mathsf{CSL}}$.
- *The minimal xor automaton for $\mathcal{L}$ is size-minimal among all mod-2-weighted automata $\mathbb{Y}$ accepting $\mathcal{L}$.*

Minimality

In the following cases closedness is non-trivial and translates to the identities below.

Corollary

- *The átomaton for $\mathcal{L}$ is size-minimal among NFA $\mathbb{Y}$ accepting $\mathcal{L}$ with* $\overline{\text{im}(\text{obs}_{\mathbb{Y}}^{\dagger})}^{\text{CSL}} = \overline{\text{im}(\text{obs}_{\mathbb{Y}}^{\dagger})}^{\text{CABA}}$.
- *The distromaton for $\mathcal{L}$ is size-minimal among NFA $\mathbb{Y}$ accepting $\mathcal{L}$ with* $\overline{\text{im}(\text{obs}_{\mathbb{Y}}^{\dagger})}^{\text{CSL}} = \overline{\text{im}(\text{obs}_{\mathbb{Y}}^{\dagger})}^{\text{CDL}}$.
- *The minimal xor-CABA automaton for $\mathcal{L}$ is size-minimal among mod-2-weighted automata $\mathbb{Y}$ accepting $\mathcal{L}$ with* $\overline{\text{im}(\text{obs}_{\mathbb{Y}}^{\dagger})}^{\mathbb{Z}_2\text{-Vect}} = \overline{\text{im}(\text{obs}_{\mathbb{Y}}^{\dagger})}^{\text{CABA}}$

Minimality

The previous results allow us to compare the size of different acceptors.

Corollary

- *If* $\overline{\mathsf{Der}(\mathcal{L})}^{\mathbb{Z}_2\text{-}\mathsf{Vect}} = \overline{\mathsf{Der}(\mathcal{L})}^{\mathsf{CABA}}$*, then the minimal xor automaton and the minimal xor-CABA automaton for* $\mathcal{L}$ *are of the same size.*
- *If* $\overline{\mathsf{Der}(\mathcal{L})}^{\mathsf{CSL}} = \overline{\mathsf{Der}(\mathcal{L})}^{\mathsf{CDL}}$*, then the canonical RFSA and the distromaton for* $\mathcal{L}$ *are of the same size.*
- *If* $\overline{\mathsf{Der}(\mathcal{L})}^{\mathsf{CSL}} = \overline{\mathsf{Der}(\mathcal{L})}^{\mathsf{CABA}}$*, then the canonical RFSA and the átomaton for* $\mathcal{L}$ *are of the same size.*

Future work

Ideas

Some rough thoughts for future work:

- Cover the canonical probabilistic RFSA[18] and canonical alternating RFSA[19];
- Utilise distributive laws between two different categories (e.g. automata product);
- Generalise Brzozowski[20] inspired double reversal characterisations.

[18] Esposito et al. 2002.

[19] Berndt et al. 2017.

[20] Brzozowski 1962.

The end

Thanks for listening!

[20] https://arxiv.org/abs/2104.13421