Generators and Bases for Monadic Closures

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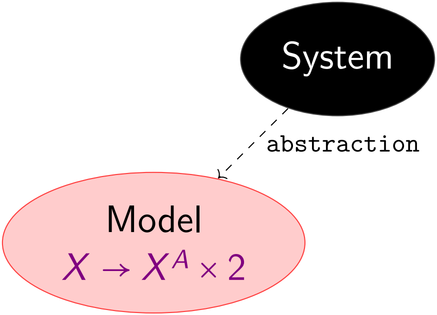
https://arxiv.org/abs/2010.10223

https://fgh.xyz

Introduction

Introduction: Automata Learning



$$[\![\text{System}]\!] \overset{\supseteq}{=} [\![\text{Model}]\!] \in A^* \to 2$$

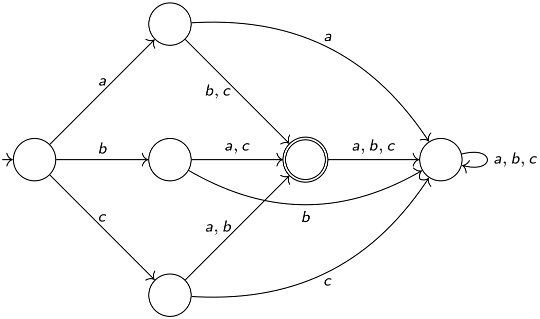
Theorem

If L^* is instantiated with $[\![\mathscr{X}]\!]$, then it terminates with the unique size-minimal DFA $m(\mathscr{X})$.

Angluin. *Learning Regular Sets from Queries and Counterexamples* (1987).

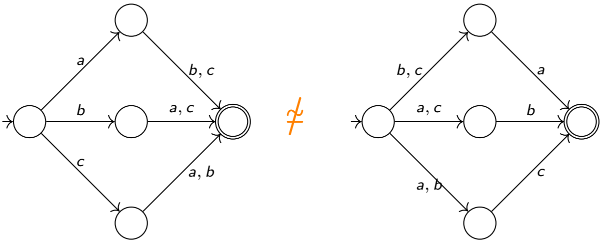
Introduction: The Canonical DFA

$\{ab, ac, ba, bc, ca, cb\} \subseteq \{a, b, c\}^*$



Introduction: Non-isomorphic Size-Minimal NFAs

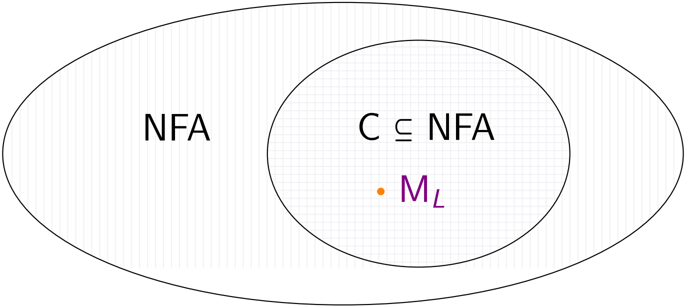
$\{ab, ac, ba, bc, ca, cb\} \subseteq \{a, b, c\}^*$



Arnold, Dicky, and Nivat. *A Note About Minimal Non-Deterministic Automata* (1992).

Introduction: Canonical Automata

Canonical RFSA[1], Distromaton[2], Átomaton[3], ...



[1] Denis, Lemay, and Terlutte. *Residual Finite State Automata* (2001).

[2] Myers, Adamek, Milius, and Urbat. *Coalgebraic Constructions of Canonical Nondeterministic Automata* (2015).

[3] Brzozowski and Tamm. *Theory of Átomata* (2014).

Introduction: General Canonical Automata

$$X \to X^A \times B$$

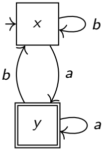
$$\downarrow$$

$$Y \to T(Y)^A \times B$$

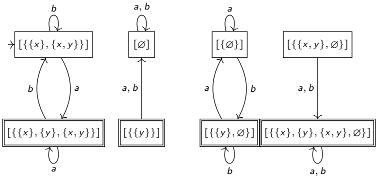
$$B, X \in \mathrm{Alg}(S)$$

Zetzsche, Heerdt, Silva, and Sammartino. *Canonical Automata via Distributive Law Homomorphisms* (2021).
Rutten, Bonsangue, Bonchi, and Silva. *Generalizing Determinization from Automata to Coalgebras* (2013).
Arbib and Manes. *Fuzzy Machines in a Category* (1975).

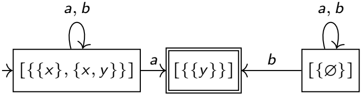
$$(a+b)^*a \subseteq \{a,b\}^*$$



(a) Minimal Coalgebra



(b) Minimal Bialgebra



(c) Canonical Automaton

Step 1: (a) → (b)   Step 2: (b) → (c)

Contributions

Contributions

Step 1

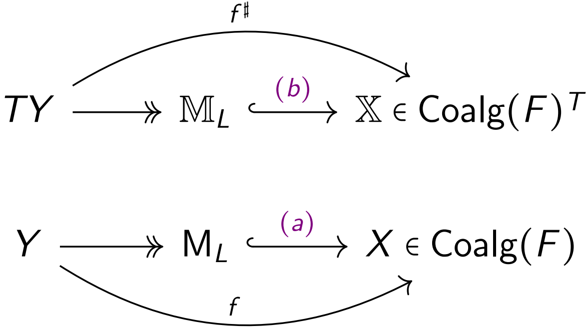
- Generalise closure of a subset of an algebra
- Functor between subobjects rel. to fact. system
- Extend functor to a monad
- Close subobjects that arise as image
- Recover minimal coalgebra to minimal bialgebra

Step 2

- Define category of algebras with generator/basis
- Adjunction to Eilenberg-Moore, monoidal
- Generalise matrix representation theory of vector spaces
- Bases for bialgebras, bases as coalgebras
- Finitary varieties, locally finitely presentable categories

Step 1: Monadic Closures

Step 1: (a) → (b)

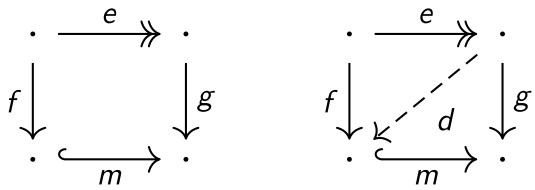


(a) Minimal Coalgebra

(b) Minimal Bialgebra

For example, $TY = \mathcal{P}Y$, $FX = 2 \times X^A$, $Y = A^*$, $X = 2^{A^*}$, and $f(w)(v) = L(wv)$, where $L : A^* \to 2$.

$(\mathscr{E}, \mathscr{M})$-Factorisation System for $\mathscr{C}$

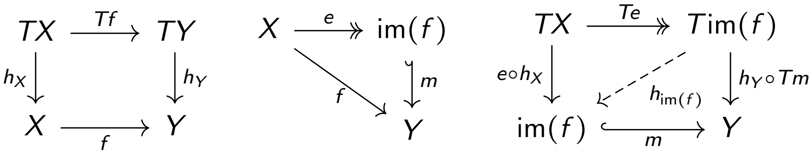


Each of $\mathscr{E}$ and $\mathscr{M}$ is closed under composition with isomorphisms.

Each morphism f in $\mathscr{C}$ can be factored as $f = m \circ e$, with $e \in \mathscr{E}$ and $m \in \mathscr{M}$.

Contributions: Step 1: Monadic Closures

$(\mathscr{E}, \mathscr{M})$-Factorisation System for $\mathscr{C}^T$

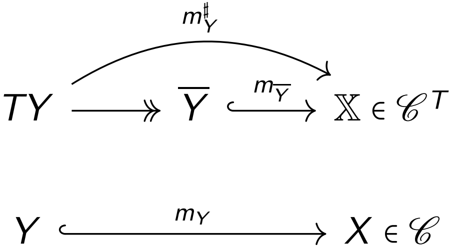


We assume that $(T : \mathscr{C} \to \mathscr{C}, \eta, \mu)$ preserves $\mathscr{E}$.

Thorsten Wißmann. *Minimality Notions via Factorization Systems and Examples* (2022).

Contributions: Step 1: Monadic Closures

$$\overline{(\cdot)}^{\mathbb{X}} : \mathsf{Sub}_{\mathscr{C}}(X) \to \mathsf{Sub}_{\mathscr{C}^T}(\mathbb{X}),\ m_Y \mapsto m_{\overline{Y}}$$

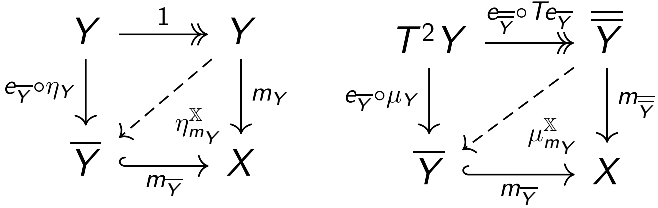


We assume that $(T : \mathscr{C} \to \mathscr{C}, \eta, \mu)$ preserves $\mathscr{E}$.

One can show that $\overline{m_{\mathrm{im}(f)}}^{\mathbb{X}} = m_{\mathrm{im}(f\sharp)}$ in $\mathsf{Sub}_{\mathscr{C}^T}(\mathbb{X})$.

Contributions: Step 1: Monadic Closures

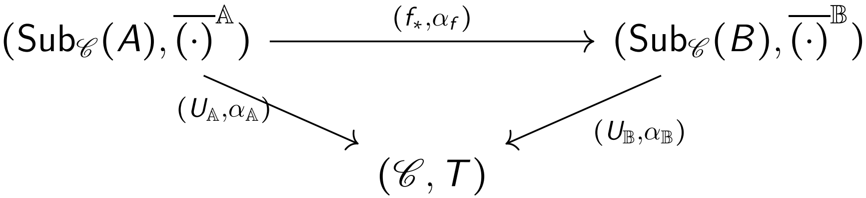
$$(\overline{(\cdot)}^{\mathbb{X}} : \mathsf{Sub}_{\mathscr{C}}(X) \to \mathsf{Sub}_{\mathscr{C}}(X), \eta^{\mathbb{X}}, \mu^{\mathbb{X}})$$



We assume that $(T : \mathscr{C} \to \mathscr{C}, \eta, \mu)$ preserves $\mathscr{E}$.

Contributions: Step 1: Monadic Closures

Compositionality



We assume that $(A : \mathscr{C} \to \mathscr{C}, \eta_A, \mu_A)$ and $(B : \mathscr{C} \to \mathscr{C}, \eta_B, \mu_B)$ preserve $\mathscr{E}$.

The functor $f_* : \mathsf{Sub}(A) \to \mathsf{Sub}(B)$ is defined by $f_*(m_X) = f \circ m_X$ and $f_*(g) = g$, where $f : \mathbb{A} \to \mathbb{B} \in \mathscr{M}$.

The functor $U_{\mathbb{X}} : \mathsf{Sub}_{\mathscr{C}}(X) \to \mathscr{C}$ is defined by $U_{\mathbb{X}}(m_Y) = Y$ and $U_{\mathbb{X}}(f) = f$.

Step 2: *Generating* Monadic Closures

Contributions: Step 2: *Generating* Monadic Closures

Step 2: (b) → (c)

$$
\begin{array}{ccc}
X & \xrightarrow{\mathrm{id}_X} & X \\
{\scriptstyle d}\downarrow & & \uparrow{\scriptstyle h} \\
TY & \xrightarrow[Ti]{} & TX
\end{array}
\qquad\qquad
\begin{array}{ccc}
TY & \xrightarrow{\mathrm{id}_{TY}} & TY \\
{\scriptstyle Ti}\downarrow & & \uparrow{\scriptstyle d} \\
TX & \xrightarrow[h]{} & X
\end{array}
$$

(b) Minimal Bialgebra

(c) Canonical Automaton

(Y, i, d) is a *generator* (*basis*) for the T-algebra (X, h).

Arbib and Manes. *Fuzzy Machines in a Category* (1975).

Zetzsche, Heerdt, Silva, and Sammartino. *Canonical Automata via Distributive Law Homomorphisms* (2021).

Contributions: Step 2: Generating Monadic Closures

$$(Y, i, d) \text{ generates } (X, \vee^h) \simeq (X, h) \in \mathsf{Set}^{\mathcal{P}}$$

iff

$$x = \bigvee_{y \in d(x)}^{h} i(y) \text{ forall } x \in X$$

Contributions: Step 2: Generating Monadic Closures

$$i^\sharp : \exp_T(Y, Fd \circ k \circ i) \longrightarrow (X, h, k)$$

Zetzsche, Heerdt, Silva, and Sammartino. *Canonical Automata via Distributive Law Homomorphisms* (2021).

Defining $\exp_T(X, k) := (TX, \mu_X, (F\mu_X \circ \lambda_{TX}) \circ Tk)$ and $\exp_T(f) := Tf$ yields a functor $\exp_T : \mathrm{Coalg}(FT) \to \mathrm{Bialg}(\lambda)$.

We assume that (Y, i, d) is a generator for the T-algebra (X, h).

Contributions: Step 2: Generating Monadic Closures

Categorification

$$F \dashv U \colon \mathrm{GAlg}(T) \leftrightarrows \mathscr{C}^T$$

GAlg(T) is monoidal, if T is

The objects of GAlg(T) are tuples $(\mathbb{X}_\alpha, \alpha)$, where $\mathbb{X}_\alpha$ is generated by α. A morphism $(f, p) : (\mathbb{X}_\alpha, \alpha) \to (\mathbb{X}_\beta, \beta)$ is a tuple $(f : \mathbb{X}_\alpha \to \mathbb{X}_\beta, p : Y_\alpha \to TY_\beta)$, s.t. $d_\beta \circ f = p^\sharp \circ d_\alpha$ and $i^\sharp_\beta \circ p^\sharp = f \circ i^\sharp_\alpha$.

$F : \mathscr{C}^T \to \mathrm{GAlg}(T)$ is defined by $F(\mathbb{X}) := (\mathbb{X}, (X, \mathrm{id}_X, \eta_X))$ and $F(f : \mathbb{X} \to \mathbb{Y}) := (f, \eta_Y \circ f)$.

$U : \mathrm{GAlg}(T) \to \mathscr{C}^T$ is defined by $U(\mathbb{X}_\alpha, \alpha) := \mathbb{X}_\alpha$ and $U(f, p) = f$.

Kleisli Representation Theory

$$f_{\alpha\beta} := Y_\alpha \xrightarrow{i_\alpha} X_\alpha \xrightarrow{f} X_\beta \xrightarrow{d_\beta} TY_\beta$$

$$p^{\alpha\beta} := X_\alpha \xrightarrow{d_\alpha} TY_\alpha \xrightarrow{Tp} T^2Y_\beta \xrightarrow{\mu_{Y_\beta}} TY_\beta \xrightarrow{Ti_\beta} TX_\beta \xrightarrow{h_\beta} X_\beta$$

$p^{\alpha\beta} : X_\alpha \to X_\beta$ is a T-algebra homomorphism $p^{\alpha\beta} : \mathbb{X}_\alpha \to \mathbb{X}_\beta$.

The operations $f \mapsto f^{\alpha\beta}$ and $p \mapsto p^{\alpha\beta}$ are mutually inverse.

Both operations are compositional, $g_{\beta\gamma} \cdot f_{\alpha\beta} = (g \circ f)_{\alpha\gamma}$ and $q^{\beta\gamma} \circ p^{\alpha\beta} = (q \cdot p)^{\alpha\gamma}$.

There exist Kleisli isomorphisms p and q such that $f_{\alpha'\beta'} = q \cdot f_{\alpha\beta} \cdot p$.

Bases for Bialgebras

$$
\begin{array}{ccc}
Y & \xrightarrow{i} & X \\
{\scriptstyle k_Y}\downarrow & & \downarrow{\scriptstyle k} \\
FY & \xrightarrow[Fi]{} & FX
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{d} & TY \\
{\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \lambda_Y \circ Tk_Y} \\
FX & \xrightarrow[Fd]{} & FTY
\end{array}
\qquad
\begin{array}{ccc}
TY & \xrightarrow{Ti} & TX \\
{\scriptstyle d}\uparrow & & \downarrow{\scriptstyle h} \\
X & \xrightarrow[\mathrm{id}_X]{} & X
\end{array}
\qquad
\begin{array}{ccc}
TX & \xrightarrow{h} & X \\
{\scriptstyle Ti}\uparrow & & \downarrow{\scriptstyle d} \\
TY & \xrightarrow[\mathrm{id}_{TY}]{} & TY
\end{array}
$$

$\mathrm{Bialg}(\lambda) \cong \mathrm{Coalg}(F)^{T_\lambda}$

If (Y, k_Y, i, d) is a generator for (X, h, k), then $i^\sharp : \mathrm{free}_T(Y, k_Y) \to (X, h, k)$, where $\mathrm{free}_T(X, k) := (TX, \mu_X, \lambda_X \circ Tk)$ and $\mathrm{free}_T(f) := Tf$. For bases, $\mathrm{free}_T(Y, k_Y) = \exp_T(Y, Fd \circ k \circ i)$.

Let (Y, i, d) be a basis for (X, h), then $(TY, (Fd \circ k \circ i)^\sharp, i^\sharp, \eta_{TY} \circ d)$ is a generator for (X, h, k).

Contributions: Step 2: Generating Monadic Closures

Further Topics

- Bases as Coalgebras
  - Comparison with previous work by Bart Jacobs
- Signatures, Equations, and Finitary Monads
  - Characterise generators for algebras over finitary monads
- Lfp Categories, Finitely Generated Objects
  - Algebra over T is finitely generated object of $\mathscr{C}^T$ iff ...

The End

Thanks For Listening!

https://arxiv.org/abs/2010.10223

https://fgh.xyz