Generators and Bases for Monadic Closures

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Abstract

It is well-known that every regular language admits a unique minimal deterministic acceptor. Establishing an analogous result for non-deterministic acceptors is significantly more difficult, but nonetheless of great practical importance. To tackle this issue, a number of sub-classes of non-deterministic automata have been identified, all admitting canonical minimal representatives. In previous work, we have shown that such representatives can be recovered categorically in two steps. First, one constructs the minimal bialgebra accepting a given regular language, by closing the minimal coalgebra with additional algebraic structure over a monad. Second, one identifies canonical generators for the algebraic part of the bialgebra, to derive an equivalent coalgebra with side effects in a monad. In this paper, we further develop the general theory underlying these two steps. On the one hand, we show that deriving a minimal bialgebra from a minimal coalgebra can be realized by applying a monad on an appropriate category of subobjects. On the other hand, we explore the abstract theory of generators and bases for algebras over a monad.

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1 Introduction

The existence of a unique minimal *deterministic* finite automaton is an important property of regular languages [29]. Establishing a similar result for *non-deterministic* finite automata is of great importance, as non-deterministic automata can be exponentially more succinct than deterministic ones, but turns out to be surprisingly difficult. The main problem is that a regular language can be accepted by several size-minimal NFAs that are not isomorphic. An example illustrating the situation is displayed in Figure 1a.

To tackle the issue, a number of sub-classes of non-deterministic automata admitting canonical representatives have been identified [15, 16, 43, 28]. One such example is the *canonical residual finite state automaton* (short *canonical RFSA*, also known as *jiromaton*), which is minimal among non-deterministic automata accepting joins of residual languages

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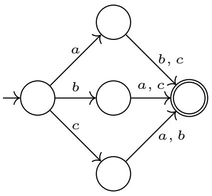


(a) Two non-isomorphic size-minimal NFA accepting $\{ab, ac, ba, bc, ca, cb\} \subseteq \{a, b, c\}^*$ [7].

$$\begin{tikzcd}
X \arrow[r, "\{-\}"] \arrow[d, "k"'] & \mathcal{P}(X) \arrow[r, dashed, "\mathrm{obs}"] \arrow[dl, "k^\sharp"] & 2^{A^*} \arrow[d, "\langle \varepsilon, \delta \rangle"] \\
2 \times \mathcal{P}(X)^A \arrow[rr, dashed, "2 \times \mathrm{obs}^A"'] & & 2 \times (2^{A^*})^A
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[d, "k"'] & TX \arrow[r, dashed, "\mathrm{obs}"] \arrow[dl, "k^\sharp"] & \Omega \arrow[d, "\omega"] \\
FTX \arrow[rr, dashed, "F\mathrm{obs}"'] & & F\Omega
\end{tikzcd} .$$

(b) Classical and generalised determinisation of automata with side-effects in a monad [35].

Figure 1 Non-isomorphic NFAs and generalised determinisation.

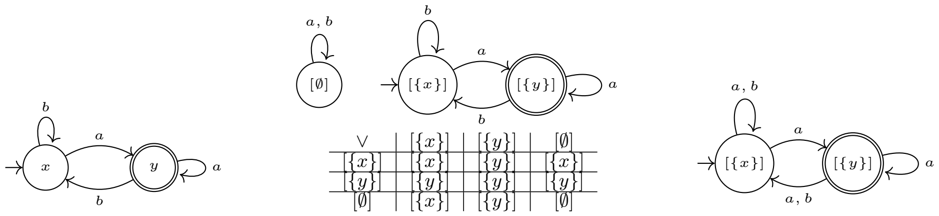
[16]. In previous work [45], we presented a categorical framework that unifies constructions and correctness proofs of canonical non-deterministic automata and unveils new ones.

The framework adopts the well-known representation of automata as coalgebras [19, 33, 32] and side-effects like non-determinism as monads [25, 26, 27]. For instance, an NFA (without initial states) is represented as a coalgebra (X, k) with side-effects in the powerset monad $(\mathcal{P}, \{-\}, \mu)$, where X is the set of states, $k\colon X \to 2 \times \mathcal{P}(X)^A$ combines the function classifying each state as accepting or rejecting with the function giving the set of next states for each input, $\{-\}$ creates singleton sets, and μ takes the union of a set of sets.

To derive canonical non-deterministic acceptors, the framework suggests a procedure that is closely related to the so-called *powerset construction*. As depicted at the top of Figure 1b, the latter converts a non-deterministic finite automaton (X, k) into an equivalent deterministic finite automaton $(\mathcal{P}X, k^\sharp)$, where $k^\sharp$ is obtained by lifting k to the subsets of X, the tuple $\langle \varepsilon, \delta \rangle$ is the automaton of languages, and the morphism obs assigns language semantics to each set of states. As seen at the bottom of Figure 1b, the construction can be generalised by replacing the functor $2 \times (-)^A$ with any (suitable) functor F describing the automaton structure, and $\mathcal{P}$ with a monad T describing the automaton side-effects, to transform a coalgebra $k : X \to FTX$ with side-effects in T into an equivalent coalgebra $k^\sharp : TX \to FTX$ [35]. Under this perspective, $\Omega \xrightarrow{\omega} F\Omega$ is the so-called *final coalgebra*, providing a semantic universe that generalises the automaton of languages. The deterministic automata resulting from such determinisation constructions have *additional algebraic structure*: the state space $\mathcal{P}(X)$ defines a free complete join-semilattice (CSL) over X and $k^\sharp$ is a CSL homomorphism. More generally, TX defines a (free) algebra for the monad T, and $k^\sharp$ is a T-algebra homomorphism, thus constituting a so-called *bialgebra* over a *distributive law* relating F and T [9, 37].

Using the powerset construction, a canonical succinct acceptor for a regular language $L \subseteq A^*$ over an alphabet A can be obtained in two steps:

1. One constructs the minimal (pointed) coalgebra M_L for the functor $F = 2 \times (-)^A$ accepting L. For the case $A = \{a, b\}$ and $L = (a + b)^* a$, the coalgebra M_L is depicted in Figure 2a. Generally, it can be obtained via the Myhill-Nerode construction [29]. One then equips the former with additional algebraic structure in a monad T (which is related to F via a canonically induced distributive law). This can be done by applying the generalised determinisation procedure to M_L, when seen as coalgebra with trivial side-effects in T. By identifying semantically equivalent states one consequently derives the minimal (pointed) bialgebra for L. If $T = \mathcal{P}$ is the powerset monad, the minimal bialgebra for the language $L = (a + b)^* a$ is depicted in Figure 2b.
2. One exploits the algebraic structure underlying the minimal bialgebra for L to "reverse" the generalised determinisation procedure. That is, one identifies a minimal set of *generators* that spans the full algebraic structure, to derive an equivalent succinct automaton with



(a) The minimal DFA **(b)** The minimal CSL-structured DFA **(c)** The canonical RFSA

Figure 2 Three automata accepting the language $(a+b)^*a \subseteq \{a,b\}^*$.

side-effects in T. For example, by choosing the *join-irreducibles*[2] for the CSL underlying the minimal bialgebra in Figure 2b as generators (in this case, the join-irreducibles are given by all non-zero states), one recovers the canonical acceptor in Figure 2c.

In this paper, we further develop the general theory underlying these two steps by making the following contributions, respectively:

First, we generalise the closure of a subset of an algebraic structure as a functor between categories of subobjects relative to a factorisation system and equip it with the structure of a monad. We investigate the closure of a particular subclass of subobjects: the ones that arise by taking the image of a morphism. We show that deriving a minimal bialgebra from a minimal coalgebra can be realized by applying the monad to a subobject in this class.

Second, we define a category of algebras with generators, which is in adjunction with the category of Eilenberg-Moore algebras, and, under certain assumptions, monoidal. We generalise the representation theory of vector spaces and discuss bases for bialgebras. We compare our ideas with an approach that generalises bases as coalgebras. We find that a basis in our sense induces a basis in their sense, and identify assumptions under which the reverse is true, too. We characterise generators for finitary varieties in the sense of universal algebra and relate our work to the theory of locally finitely presentable categories.

2 Preliminaries

We assume basic knowledge of category theory (including functors, natural transformations, adjunctions), for an overview see e.g. [8].

We briefly recall the definitions of coalgebras, monads, and Eilenberg-Moore algebras. A *coalgebra* for an endofunctor F in a category $\mathcal{C}$ is a tuple (X,k) consisting of an object X in $\mathcal{C}$ and a morphism $k\colon X \to FX$. The category of coalgebras for F is denoted by $\mathrm{Coalg}(F)$. A *monad* on a category $\mathcal{C}$ is a tuple (T,η,μ) consisting of an endofunctor $T : \mathcal{C} \to \mathcal{C}$ and natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ satisfying $\mu \circ T\mu = \mu \circ \mu_T$ and $\mu \circ \eta_T = \mathrm{id}_T = \mu \circ T\eta$. An *Eilenberg-Moore algebra* over a monad T on $\mathcal{C}$ is a tuple (X,h) consisting of an object X in $\mathcal{C}$ and a morphism $h : TX \to X$ satisfying $h \circ \mu_X = h \circ Th$ and $h \circ \eta_X = \mathrm{id}_X$. The category of Eilenberg-Moore algebras over T is denoted by $\mathrm{Alg}(T)$.

We now introduce other notions that are necessary to follow our technical development: distributive laws, bialgebras, and generators and bases for algebras over a monad.

Distributive laws have originally occurred as a way to compose monads [9], but now also exist in a wide range of other forms [37]. For our case it is sufficient to consider distributive laws between a monad and an endofunctor, sometimes called *Eilenberg-Moore laws* [20].

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[2] A *join-irreducible* is a non-zero element a satisfying, for all $y,z \in L$ with $a = y \vee z$, that $a = y$ or $a = z$.

▶ **Definition 1** (Distributive Law). *A* distributive law *between a monad T and an endofunctor F on $\mathcal{C}$ is a natural transformation $\lambda : TF \Rightarrow FT$ satisfying $F\eta_X = \lambda_X \circ \eta_{FX}$ and $\lambda_x \circ \mu_{FX} = F\mu_X \circ \lambda_{TX} \circ T\lambda_X$.*

Given a distributive law, one can model the determinisation of a system with dynamics in F and side-effects in T by lifting a FT-coalgebra (X, k) to the F-coalgebra $(TX, k^\sharp)$, where $k^\sharp := (F\mu_X \circ \lambda_{TX}) \circ Tk$. As one verifies, $k^\sharp$ is a T-algebra homomorphism of type $(TX, \mu_X) \to (FTX, F\mu_X \circ \lambda_{TX})$. There exists a distributive law for which the lifting $k^\sharp$ is the DFA in CSL obtained from an NFA k via the classical powerset construction [35].

The example illustrates the concept of a bialgebra: the algebraic part (TX, μ_X) and the coalgebraic part $(TX, k^\sharp)$ of a lifted automaton are compatible along the distributive law λ.

▶ **Definition 2** (Bialgebra). *A* λ-bialgebra *is a tuple (X, h, k) consisting of a T-algebra (X, h) and an F-coalgebra (X, k) satisfying $Fh \circ \lambda_X \circ Tk = k \circ h$.*

A homomorphism between λ-bialgebras is a morphism between the underlying objects that is simultaneously a T-algebra homomorphism and an F-coalgebra homomorphism. The category of λ-bialgebras and homomorphisms is denoted by $\mathrm{Bialg}(\lambda)$.

The generalised determinisation procedure can now be rephrased as follows.

▶ **Lemma 3** ([20]).
- *Defining* $\exp_T(X, k) := (TX, \mu_X, (F\mu_X \circ \lambda_{TX}) \circ Tk)$ *and* $\exp_T(f) := Tf$ *yields a functor* $\exp_T : \mathrm{Coalg}(FT) \to \mathrm{Bialg}(\lambda)$.
- *Defining* $\mathrm{free}_T(X, k) := (TX, \mu_X, \lambda_X \circ Tk)$ *and* $\mathrm{free}_T(f) := Tf$ *yields a functor* $\mathrm{free}_T : \mathrm{Coalg}(F) \to \mathrm{Bialg}(\lambda)$ *satisfying* $\mathrm{free}_T(X, k) = \exp_T(X, F\eta_X \circ k)$.

The last ingredient is a generalisation of generators for structures such as vector spaces.

▶ **Definition 4** (Generator and Basis [45]). *A* generator *for a T-algebra (X, h) is a tuple (Y, i, d) consisting of an object Y, a morphism $i\colon Y \to X$, and a morphism $d\colon X \to TY$ such that $(h \circ Ti) \circ d = \mathrm{id}_X$. A generator is called a* basis *if it additionally satisfies $d \circ (h \circ Ti) = \mathrm{id}_{TY}$.*

A generator for a T-algebra is called a *scoop* by Arbib and Manes [6]. Every T-algebra (X, h) is generated by $(X, \mathrm{id}_X, \eta_X)$ and admits a basis if(f) it is isomorphic to a free algebra.

▶ **Example 5.** A tuple (Y, i, d) is a generator for a $\mathcal{P}$-algebra $L = (X, h) \simeq (X, \vee^h)$ if(f) $x = \vee^h_{y \in d(x)} i(y)$ for all $x \in X$. Note that if $Y \subseteq X$ is a subset, then $i(y) = y$ for all $y \in Y$. If L satisfies the descending chain condition, which is in particular the case if X is finite, then defining $i(y) = y$ and $d(x) = \{y \in J(L) \mid y \leq x\}$ turns the set of join-irreducibles $J(L)$ into a size-minimal generator $(J(L), i, d)$ for L.

A central result in [45] shows that is enough to find generators for the underlying algebra of a bialgebra to derive an equivalent free bialgebra. This is because the algebraic and coalgebraic components are tightly intertwined via a distributive law.

▶ **Proposition 6** ([45]). *Let (X, h, k) be a λ-bialgebra and let (Y, i, d) be a generator for the T-algebra (X, h). Then $h \circ Ti\colon \exp_T(Y, Fd \circ k \circ i) \to (X, h, k)$ is a λ-bialgebra homomorphism.*

3 Step 1: Closure

In this section, we further explore the categorical construction of minimal canonical acceptors given in [45]. In particular, we show that deriving a minimal bialgebra from a minimal coalgebra by closing the latter with additional algebraic structure has a direct analogue in universal algebra: taking the closure of a subset of an algebra.

$$\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "h_X"'] & TY \arrow[d, "h_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, two heads, "e"] \arrow[dr, "f"'] & \mathrm{im}(f) \arrow[d, hook, "m"] \\
 & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
TX \arrow[r, two heads, "Te"] \arrow[d, "e \circ h_X"'] & T\mathrm{im}(f) \arrow[d, "h_Y \circ Tm"] \arrow[dl, dashed, "h_{\mathrm{im}(f)}"] \\
\mathrm{im}(f) \arrow[r, hook, "m"'] & Y
\end{tikzcd}$$

Figure 3 Factorising a T-algebra homomorphism via the factorisation system of a base category.

3.1 Factorisation Systems and Subobjects

In the category of sets and functions, every morphism can be factored into a surjection onto its image followed by an injection into its codomain. In this section we introduce a convenient abstraction of this phenomenon for arbitrary categories. The ideas are well established [13, 31, 24]. We choose to adapt the formalism of [2].

▶ **Definition 7** (Factorisation System). *Let $\mathcal{E}$ and $\mathcal{M}$ be classes of morphisms in a category $\mathcal{C}$. We call the tuple $(\mathcal{E}, \mathcal{M})$ a* factorisation system *for $\mathcal{C}$ if the following three conditions hold:*

(F1) Each of $\mathcal{E}$ and $\mathcal{M}$ is closed under composition with isomorphisms.

(F2) Each morphism f in $\mathcal{C}$ can be factored as $f = m \circ e$, with $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

(F3) Whenever $g \circ e = m \circ f$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there exists a unique diagonal d, such that $f = d \circ e$ and $g = m \circ d$.

We use double headed ($\twoheadrightarrow$) and hooked ($\hookrightarrow$) arrows to indicate that a morphism is in $\mathcal{E}$ and $\mathcal{M}$, respectively. We refer to m and e as the *mono* and *epi* parts. The codomain of e, or equivalently, the domain of m, is called the *image* of f and denoted by $\mathrm{im}(f)$.

One can show that each of $\mathcal{E}$ and $\mathcal{M}$ contains all isomorphisms and is closed under composition [2, Prop. 14.6]. From the uniqueness condition on the diagonal one can deduce that factorisations are unique up to unique isomorphism [2, Prop. 14.4]. It further follows that $\mathcal{E}$ has the *right cancellation property*, that is $g \circ f \in \mathcal{E}$ and $f \in \mathcal{E}$ implies $g \in \mathcal{E}$. Dually, $\mathcal{M}$ has the *left cancellation property*, that is, $g \circ f \in \mathcal{M}$ and $g \in \mathcal{M}$ implies $f \in \mathcal{M}$ [2, Prop. 14.9]. As stated in [2, Prop. 14.6], a factorisation system in the sense of Definition 7 induces a *weak factorisation system* in the sense of [31], that is, a slightly weaker version of what is well-known as *orthogonal factorisation system* [31].

As intended, in the category of sets and functions, surjective and injective functions, or equivalently, epi- and monomorphisms, constitute a factorisation system [2, Ex. 14.2]. More involved examples can be constructed for e.g. the category of topological spaces or the category of categories [2, Ex. 14.2]. We are particularly interested in factorisation systems for the categories of algebras over a monad and coalgebras over an endofunctor.

The naive categorification of a subset $Y \subseteq X$ is a monomorphism $Y \to X$. Since in the category of sets epi- and monomorphism constitute a factorisation system, we may generalise subsets to arbitrary categories $\mathcal{C}$ with a factorisation system $(\mathcal{E}, \mathcal{M})$ in the following way:

▶ **Definition 8** (Subobjects). *A* subobject *of an object $X \in \mathcal{C}$ is a morphism $m_Y : Y \hookrightarrow X \in \mathcal{M}$. A morphism $f : m_{Y_1} \to m_{Y_2}$ between subobjects of X consists of a morphism $f : Y_1 \to Y_2$ such that $m_{Y_2} \circ f = m_{Y_1}$.*

The category of (isomorphism classes of) subobjects of X is denoted by $\mathrm{Sub}(X)$.

Note that as $\mathcal{M}$ has the left cancellation property, every morphism between subobjects in fact lies in $\mathcal{M}$. We work with isomorphism classes of subobjects since factorisations of morphisms are only defined up to unique isomorphism.

$$\begin{tikzcd}
\mathcal{C}/X \arrow[r] & \mathrm{Alg}(T)/\mathbb{X} \arrow[d] \\
\mathrm{Sub}(X) \arrow[u] \arrow[r, "\overline{(\cdot)}^{\mathbb{X}}"] & \mathrm{Sub}(\mathbb{X})
\end{tikzcd}$$

(a) Decomposition

$$\begin{tikzcd}
\mathcal{C}/X \arrow[r] \arrow[d] & \mathrm{Alg}(T)/\mathbb{X} \arrow[d] \\
\mathrm{Sub}(X) \arrow[r, "\overline{(\cdot)}^{\mathbb{X}}"] & \mathrm{Sub}(\mathbb{X})
\end{tikzcd}$$

(b) Commutativity

Figure 4 A high-level perspective on the subobject closure functor defined in Proposition 11.

3.2 Factorising Algebra Homomorphisms

In this section, we recall that if one is given a category $\mathcal{C}$ with a factorisation system $(\mathcal{E}, \mathcal{M})$ and a monad T on $\mathcal{C}$ that preserves $\mathcal{E}$ (that is, satisfies $T(e) \in \mathcal{E}$ for all $e \in \mathcal{E}$), it is possible to lift the factorisation system of the base category $\mathcal{C}$ to a factorisation system on the category of Eilenberg-Moore algebras $\mathrm{Alg}(T)$.

The result appears in e.g [44] and may be relaxed to algebras over an endofunctor. It can also be stated in its dual version: if an endofunctor on $\mathcal{C}$ preserves $\mathcal{M}$, it is possible to lift the factorisation system of $\mathcal{C}$ to the category of coalgebras [21, 44].

The induced factorisation system for $\mathrm{Alg}(T)$ consists of those algebra homomorphisms, whose underlying morphism lies in $\mathcal{E}$ or $\mathcal{M}$, respectively. Clearly in such a system condition (F1) holds. The next result shows that it also satisfies (F3).

▶ **Lemma 9** ([44, Lem. 3.6]). *Whenever $g \circ e = m \circ f$ for T-algebra homomorphisms f, g, e, m, with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there exists a unique diagonal T-algebra homomorphism d, such that $f = d \circ e$ and $g = m \circ d$.*

Let us now show that the proposed factorisation system satisfies (F2). Assume we are given a homomorphism f as on the left of Figure 3. Using the factorisation system of the base category $\mathcal{C}$, we can factorise it, as ordinary morphism, into $e \in \mathcal{E}$ and $m \in \mathcal{M}$. In consequence the outer square of the diagram on the right of Figure 3 commutes. Since by assumption the morphism Te is again in $\mathcal{E}$, we thus find a unique diagonal $h_{\mathrm{im}(f)}$ in $\mathcal{C}$ that makes the triangles on the right of Figure 3 commute. The result below shows that $h_{\mathrm{im}(f)}$ equips $\mathrm{im}(f)$ with the structure of a T-algebra.

▶ **Lemma 10** ([44, Prop. 3.7]). *$(\mathrm{im}(f), h_{\mathrm{im}(f)})$ is an Eilenberg-Moore T-algebra.*

We thus obtain an epi-mono factorisation of $f : (X, h_X) \to (Y, h_Y)$ into Eilenberg-Moore T-algebra homomorphisms $e : (X, h_X) \twoheadrightarrow (\mathrm{im}(f), h_{\mathrm{im}(f)})$ and $m : (\mathrm{im}(f), h_{\mathrm{im}(f)}) \hookrightarrow (Y, h_Y)$.

3.3 The Subobject Closure Functor

While subobjects in the category of sets generalise subsets, subobjects in the category of algebras generalise subalgebras. By taking the algebraic closure of a subset of an algebra one can thus transition from one category of subobjects to the other.

In this section, we generalise this phenomenon from the category of sets to more general categories. As before, we assume a base category $\mathcal{C}$ with a factorisation system $(\mathcal{E}, \mathcal{M})$ and a monad T on $\mathcal{C}$ that preserves $\mathcal{E}$. Our aim is to construct, for any T-algebra $\mathbb{X}$ with carrier X, a functor from the subobjects $\mathrm{Sub}(X)$ in $\mathcal{C}$ to the subobjects $\mathrm{Sub}(\mathbb{X})$ in $\mathrm{Alg}(T)$.

Recall the free Eilenberg-Moore algebra adjunction. For any object Y in $\mathcal{C}$ and T-algebra $\mathbb{X} = (X, h)$, it maps a morphism $\varphi : Y \to X$ to the T-algebra homomorphism $\varphi^\sharp := h \circ T\varphi : (TY, \mu_Y) \to \mathbb{X}$. In Section 3.1 we have seen that the factorisation system of $\mathcal{C}$ naturally lifts to a factorisation system on the category of T-algebras. In particular, we know that up to isomorphism the homomorphism $\varphi^\sharp$ admits a factorisation into algebra homomorphisms of the form $\varphi^\sharp = m_{\mathrm{im}(\varphi^\sharp)} \circ e_{\mathrm{im}(\varphi^\sharp)}$. If the morphism φ is given by a subobject m_Y, let $\overline{Y} := (\mathrm{im}(m_Y^\sharp), h_{\mathrm{im}(m_Y^\sharp)})$, then above construction yields a second subobject $m_{\overline{Y}}$:

$$m_Y : Y \to X \in \mathcal{M} \qquad\qquad m_{\overline{Y}} : \overline{Y} \to \mathbb{X} \in \mathcal{M}.$$

Since for any morphism $f : m_{Y_1} \to m_{Y_2}$ between subobjects of X the outer square of (1) commutes, there exists a unique diagonal algebra homomorphism $\overline{f} : m_{\overline{Y_1}} \to m_{\overline{Y_2}}$ between subobjects of $\mathbb{X}$ making the two triangles in (1) below commute:

$$\begin{tikzcd}
(TY_1, \mu_{Y_1}) \arrow[r, two heads, "e_{\overline{Y_1}}"] \arrow[d, "e_{\overline{Y_2}} \circ Tf"'] & (\overline{Y_1}, h_{\overline{Y_1}}) \arrow[d, "m_{\overline{Y_1}}"] \arrow[dl, dashed, "\overline{f}"] \\
(\overline{Y_2}, h_{\overline{Y_2}}) \arrow[r, hook, "m_{\overline{Y_2}}"'] & (X, h)
\end{tikzcd} . \tag{1}$$

The following result shows that above constructions are compositional.

▶ **Proposition 11.** *Assigning $m_Y \mapsto m_{\overline{Y}}$ and $f \mapsto \overline{f}$ yields a functor $\overline{(\cdot)}^{\mathbb{X}} : \mathrm{Sub}(X) \to \mathrm{Sub}(\mathbb{X})$.*

Mapping an algebra homomorphism with codomain $\mathbb{X}$ to the mono part of its epi-mono factorisation extends to a functor from the slice category over $\mathbb{X}$ to the category of subobjects of $\mathbb{X}$. Similarly, one observes that the free Eilenberg-Moore algebra adjunction gives rise to a functor from the slice category over X to the slice category over $\mathbb{X}$. Finally, it is clear that the category of subobjects of X canonically embeds into the slice category over X. The functor defined in Proposition 11 can thus be recognised as the composition in Figure 4a.

3.4 The Subobject Closure Monad

In this section, we show that the functor in Proposition 11 induces a monad on the category of subobjects $\mathrm{Sub}(X)$. As before, we assume a base category $\mathcal{C}$ with a factorisation system $(\mathcal{E}, \mathcal{M})$ and a monad $T = (T, \eta, \mu)$ on $\mathcal{C}$ that preserves $\mathcal{E}$.

We begin by establishing the following two technical identities, which assume a T-algebra $\mathbb{X} = (X, h)$ and a subobject $m_Y : Y \to X \in \mathcal{M}$.

▶ **Lemma 12.** *The following two equalities hold:*

- $m_{\overline{Y}} \circ e_{\overline{Y}} \circ \eta_Y = m_Y$
- $m_{\overline{Y}} \circ e_{\overline{Y}} \circ \mu_Y = m_{\overline{\overline{Y}}} \circ e_{\overline{\overline{Y}}} \circ Te_{\overline{Y}}$

In consequence, we can define candidates for the monad unit $\eta^{\mathbb{X}}$ and the monad multiplication $\mu^{\mathbb{X}}$, respectively, as the unique diagonals in Figure 5a. By construction both morphisms are homomorphisms of subobjects:

$$\eta^{\mathbb{X}}_{m_Y} : m_Y \longrightarrow m_{\overline{Y}} \qquad \mu^{\mathbb{X}}_{m_Y} : m_{\overline{\overline{Y}}} \longrightarrow m_{\overline{Y}}.$$

The remaining proofs of naturality and the monad laws are covered below. By a slight abuse of notation, we write $\overline{(\cdot)}^{\mathbb{X}}$ for the endofunctor on $\mathrm{Sub}(X)$ that arises by post-composition of the functor in Proposition 11 with the canonical forgetful functor from $\mathrm{Sub}(\mathbb{X})$ to $\mathrm{Sub}(X)$.

▶ **Theorem 13.** *$(\overline{(\cdot)}^{\mathbb{X}}, \eta^{\mathbb{X}}, \mu^{\mathbb{X}})$ is a monad on $\mathrm{Sub}(X)$.*

We will now show that the assignment which maps an algebra $\mathbb{X}$ to the monad $\overline{(\cdot)}^{\mathbb{X}}$ in Theorem 13 extends to algebra homomorphisms. To this end, for any algebra homomorphism $f : \mathbb{A} \to \mathbb{B}$ in $\mathcal{M}$, let $f_* : \mathrm{Sub}(A) \to \mathrm{Sub}(B)$ be the induced functor defined by $f_*(m_X) = f \circ m_X$ and $f_*(g) = g$. The result below shows that f_* can be extended to a morphism (in the sense of [36]) between monads.

$$\begin{tikzcd}
Y \arrow[r, two heads, "1"] \arrow[d, "e_{\overline{Y}} \circ \eta_Y"'] & Y \arrow[d, "m_Y"] \arrow[dl, dashed, "\eta^{\mathbb{X}}_{m_Y}"] \\
\overline{Y} \arrow[r, hook, "m_{\overline{Y}}"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2Y \arrow[r, two heads, "e_{\overline{\overline{Y}}} \circ T e_{\overline{Y}}"] \arrow[d, "e_{\overline{Y}} \circ \mu_Y"'] & \overline{\overline{Y}} \arrow[d, "m_{\overline{\overline{Y}}}"] \arrow[dl, dashed, "\mu^{\mathbb{X}}_{m_Y}"] \\
\overline{Y} \arrow[r, hook, "m_{\overline{Y}}"'] & X
\end{tikzcd}$$

(a) Induced unit $\eta^{\mathbb{X}}$ and multiplication $\mu^{\mathbb{X}}$ of the monad in Theorem 13.

$$\begin{tikzcd}
\overline{(\cdot)}^{\mathbb{A}} \arrow[rr, "{(f_*, \alpha_f)}"] \arrow[dr, "{(U_{\mathbb{A}}, \alpha_{\mathbb{A}})}"'] & & \overline{(\cdot)}^{\mathbb{B}} \arrow[dl, "{(U_{\mathbb{B}}, \alpha_{\mathbb{B}})}"] \\
& T &
\end{tikzcd}$$

(b) Commutativity of the monad morphisms in Lemma 14 and Lemma 15.

Figure 5 Structure and properties of the monad in Theorem 13.

▶ **Lemma 14.** *For any* $f : \mathbb{A} \to \mathbb{B} \in \mathcal{M}$, *there exists a monad morphism* $(f_*, \alpha) : \overline{(\cdot)}^{\mathbb{A}} \to \overline{(\cdot)}^{\mathbb{B}}$.

The next statement establishes that the canonical forgetful functor $U : \mathrm{Sub}(X) \to \mathcal{C}$ defined by $U(m_Y) = Y$ and $U(f) = f$ extends to a morphism between monads.

▶ **Lemma 15.** *There exists a monad morphism* $(U, \alpha) : \overline{(\cdot)}^{\mathbb{X}} \to T$.

We conclude with the observation that the monad morphism defined in Lemma 14 commutes with the monad morphisms defined in Lemma 15.

▶ **Lemma 16.** *Figure 5b commutes for any algebra homomorphism* $f : \mathbb{A} \to \mathbb{B} \in \mathcal{M}$.

3.5 Closing an Image

In this section we investigate the closure of a particular class of subobjects: the ones that arise by taking the image of a morphism. We then show that deriving a minimal bialgebra from a minimal coalgebra by equipping the latter with additional algebraic structure can be realized as the closure of a subobject in this class.

As before, we assume a category $\mathcal{C}$ with a factorisation system $(\mathcal{E}, \mathcal{M})$ and a monad T on $\mathcal{C}$ that preserves $\mathcal{E}$. Let also be given a T-algebra $\mathbb{X} = (X, h_X)$ and a morphism $f : Y \to X$ in $\mathcal{C}$. On the one hand, there exists a factorisation of f in $\mathcal{C}$:

$$f = Y \overset{e_{\mathrm{im}(f)}}{\twoheadrightarrow} \mathrm{im}(f) \overset{m_{\mathrm{im}(f)}}{\hookrightarrow} X.$$

On the other hand, there exists a factorisation of the lifing $f^\sharp = h_X \circ Tf$ in the category of Eilenberg-Moore algebras $\mathrm{Alg}(T)$:

$$f^\sharp = (TY, \mu_Y) \overset{e_{\mathrm{im}(f^\sharp)}}{\twoheadrightarrow} (\mathrm{im}(f^\sharp), h_{\mathrm{im}(f^\sharp)}) \overset{m_{\mathrm{im}(f^\sharp)}}{\hookrightarrow} (X, h_X).$$

The next result shows that, up to isomorphism, the closure of the subobject $m_{\mathrm{im}(f)}$ with respect to the algebra $\mathbb{X}$ is given by the subobject $m_{\mathrm{im}(f^\sharp)}$.

▶ **Lemma 17.** $\overline{m_{\mathrm{im}(f)}}^{\mathbb{X}} = m_{\mathrm{im}(f^\sharp)}$ *in* $\mathrm{Sub}(\mathbb{X})$.

The following example uses Lemma 17 to show that deriving a minimal bialgebra from a minimal coalgebra can be realised by applying a monad of the type in Theorem 13.

▶ **Example 18** (**Closure of Minimal Coalgebras**)**.** Let F be the set endofunctor with $FX = B \times X^A$, for fixed sets A and B. As F preserves monomorphisms, the canonical epi-mono factorisation system of the category of sets lifts to the category $\mathrm{Coalg}(F)$, which consists of unpointed Moore automata with input A and output B.

For any language $L : A^* \to B$, there exists a size-minimal Moore automaton M_L that accepts L. It can be recovered as the epi-mono factorisation of the final F-coalgebra homomorphism $\mathrm{obs} : A^* \to \Omega$, where Ω is carried by B^{A^*} [40], that is, $M_L = m_{\mathrm{im}(\mathrm{obs})}$.

As seen in [45], any algebra structure $h : TB \to B$ over a set monad T induces a distributive law λ between T and F. It is well-known [39] that λ-bialgebras are algebras over the monad T_λ on $\mathrm{Coalg}(F)$ defined by $T_\lambda(X, k) = (TX, \lambda_X \circ Tk)$ and $T_\lambda f = Tf$. One such T_λ-algebra is the final F-coalgebra Ω, when equipped with a canonical algebra structure.

The functor T_λ preserves epimorphisms in the category $\mathrm{Coalg}(F)$, if T preserves epimorphisms in the category of sets. The latter is the case for every set functor. By Theorem 13, there thus exists a well-defined monad $\overline{(\cdot)}^{\Omega}$ on $\mathrm{Sub}(\Omega)$.

By construction, the minimal Moore automaton M_L lives in $\mathrm{Sub}(\Omega)$. Reviewing the constructions in [45] shows that the minimal λ-bialgebra $\mathcal{M}_L$ for L is given by the image of the lifting of obs, that is, $\mathcal{M}_L = m_{\mathrm{im}(\mathrm{obs}^\sharp)}$. From Lemma 17 it thus follows $\mathcal{M}_L = \overline{M_L}^{\Omega}$. In other words, the minimal λ-bialgebra for L can be obtained from the minimal F-coalgebra for L by closing the latter with respect to the T_λ-algebra structure of Ω.

For an example of the unit η^Ω, observe how the minimal coalgebra in Figure 2a embeds into the minimal bialgebra in Figure 2b. The situation can be further generalised to arbitrary set endofunctors F that preserve monomorphisms and monads T that distribute over F.

Finally, using analogous functors to the ones present in Figure 4a, we observe that, as a consequence of Lemma 17, the diagram in Figure 4b commutes.

4 Step 2: Generators and Bases

One of the central notions of linear algebra is the *basis*: a subset of a vector space is called basis, if every vector can be uniquely written as a linear combination of basis elements.

Part of the importance of bases stems from the convenient consequences that follow from their existence. For example, linear transformations between vector spaces admit matrix representations relative to pairs of bases [22], which can be used for efficient calculations. The idea of a basis however is not restricted to the theory of vector spaces: other algebraic theories have analogous notions of bases – and generators, by waiving the uniqueness constraint –, for instance modules, semi-lattices, Boolean algebras, convex sets, and many more. In fact, the theory of bases for vector spaces is special only in the sense that every vector space admits a basis, which is not the case for e.g. modules.

In this section, we use the abstraction of generators and bases given in Definition 4 to lift results from one theory to the others. For example, one may wonder if there exists a matrix representation theory for convex sets that is analogous to the one of vector spaces.

4.1 Categorification

This section extends the definitions of generators and bases with a notion of morphism.

▶ **Definition 19** ($\mathrm{GAlg}(T)$)**.** *The category* $\mathrm{GAlg}(T)$ *of algebras with a generator over a monad T is defined as follows: objects are pairs $(\mathbb{X}_\alpha, \alpha)$, where $\mathbb{X}_\alpha = (X_\alpha, h_\alpha)$ is a T-algebra with generator $\alpha = (Y_\alpha, i_\alpha, d_\alpha)$; a morphism $(f, p) : (\mathbb{X}_\alpha, \alpha) \to (\mathbb{X}_\beta, \beta)$ consists of a T-algebra homomorphism $f : \mathbb{X}_\alpha \to \mathbb{X}_\beta$ and a Kleisli-morphism $p : Y_\alpha \to TY_\beta$, such that the diagram below commutes:*

$$\begin{tikzcd}
X_\alpha \arrow[r, "d_\alpha"] \arrow[d, "f"'] & TY_\alpha \arrow[r, "i_\alpha^\sharp"] \arrow[d, "p^\sharp"] & X_\alpha \arrow[d, "f"] \\
X_\beta \arrow[r, "d_\beta"] & TY_\beta \arrow[r, "i_\beta^\sharp"] & X_\beta
\end{tikzcd} . \tag{2}$$

The composition of morphisms is defined componentwise as $(g, q) \circ (f, p) := (g \circ f, q \cdot p)$, where $q \cdot p := \mu_{Y_\gamma} \circ Tq \circ p$ denotes the usual Kleisli-composition.

The category $\mathrm{BAlg}(T)$ of algebras with a basis is defined as full subcategory of $\mathrm{GAlg}(T)$. The category of Eilenberg-Moore T-algebras and $\mathrm{GAlg}(T)$ are in an adjoint relation.

▶ **Lemma 20.** *There exists a free-forgetful adjunction* $\mathrm{GAlg}(T) \dashv \mathrm{Alg}(T)$.

4.2 Products

In this section we show that, under certain assumptions, the monoidal product of a category naturally extends to a monoidal product of algebras with bases within that category. As a natural example we obtain the tensor-product of vector spaces with fixed bases.

We assume basic familiarity with monoidal categories. A monoidal monad T on a monoidal category $(\mathcal{C}, \otimes, I)$ is a monad which is equipped with natural transformations $T_{X,Y} : TX \otimes TY \to T(X \otimes Y)$ and $T_0 : I \to TI$, satisfying certain coherence conditions (see e.g. [34]). One can show that, given such additional data, the monoidal structure of $\mathcal{C}$ induces a monoidal category $(\mathrm{Alg}(T), \boxtimes, (TI, \mu_I))$, if two appropriately defined[3] assumptions (A1) and (A2) are satisfied [34, Corollary 2.5.6]. The two monoidal products $\otimes$ and $\boxtimes$ are related via the natural embedding $q_{\mathbb{X}_\alpha,\mathbb{X}_\beta} \circ \eta_{X_\alpha \otimes X_\beta}$, in the following referred to as $\iota_{\mathbb{X}_\alpha,\mathbb{X}_\beta}$. One can prove that the product $TY_\alpha \boxtimes TY_\beta$ is given by $T(Y_\alpha \otimes Y_\beta)$ and the coequaliser q_{TY_α,TY_β} by $\mu_{Y_\alpha \otimes Y_\beta} \circ T(T_{Y_\alpha,Y_\beta})$, where we abbreviate the free algebra (TY, μ_Y) as TY [34].

With the previous remarks in mind, we are able to claim the following.

▶ **Lemma 21.** *Let T be a monoidal monad on $(\mathcal{C}, \otimes, I)$ satisfying (A1) and (A2). Let $\alpha = (Y_\alpha, i_\alpha, d_\alpha)$ and $\beta = (Y_\beta, i_\beta, d_\beta)$ be generators (bases) for T-algebras $\mathbb{X}_\alpha$ and $\mathbb{X}_\beta$. Then $\alpha \boxtimes \beta = (Y_\alpha \otimes Y_\beta, \iota_{\mathbb{X}_\alpha,\mathbb{X}_\beta} \circ (i_\alpha \otimes i_\beta), (d_\alpha \boxtimes d_\beta))$ is a generator (basis) for the T-algebra $\mathbb{X}_\alpha \boxtimes \mathbb{X}_\beta$.*

▶ **Corollary 22.** *Let T be a monoidal monad on $(\mathcal{C}, \otimes, I)$ such that (A1) and (A2) are satisfied. The definitions $(\mathbb{X}_\alpha, \alpha) \boxtimes (\mathbb{X}_\beta, \beta) := (\mathbb{X}_\alpha \boxtimes \mathbb{X}_\beta, \alpha \boxtimes \beta)$ and $(f, p) \boxtimes (g, q) := (f \boxtimes g, T_{Y_{\alpha'},Y_{\beta'}} \circ (p \otimes q))$ yield monoidal structures with unit $((TI, \mu_I), (I, \eta_I, \mathrm{id}_{TI}))$ on* $\mathrm{GAlg}(T)$ *and* $\mathrm{BAlg}(T)$.

We conclude by instantiating above construction to the setting of vector spaces.

▶ **Example 23** (Tensor Product of Vector Spaces). Recall the free $\mathbb{K}$-vector space monad $\mathcal{V}_\mathbb{K}$ defined by $\mathcal{V}_\mathbb{K}(X) = X \to \mathbb{K}$ and $\mathcal{V}_\mathbb{K}(\varphi)(y) = \sum_{x \in f^{-1}(y)} \varphi(x)$. Its unit is given by $\eta_X(x)(y) = [x = y]$ and its multiplication by $\mu_X(\Phi)(x) = \sum_{\varphi \in \mathbb{K}^X} \Phi(\varphi) \cdot \varphi(x)$.

The category of sets is monoidal (in fact, cartesian) with respect to the cartesian product $\times$ and the singleton set $\{\star\}$. The monad $\mathcal{V}_\mathbb{K}$ is monoidal when equipped with $(\mathcal{V}_\mathbb{K})_{X,Y}(\varphi, \psi)(x, y) := \varphi(x) \cdot \psi(y)$ and $T_0(\star)(\star) := 1_\mathbb{K}$ [30]. The category of $\mathcal{V}_\mathbb{K}$-algebras is isomorphic to the category of $\mathbb{K}$-vector spaces, and satisfies (A1) and (A2). The monoidal structure induced by $\mathcal{V}_\mathbb{K}$ is the usual tensor product $\otimes$ with the unit field $\mathcal{V}_\mathbb{K}(\{\star\}) \simeq \mathbb{K}$.

Lemma 21 captures the well-known fact that the dimension of the tensor product of two vector spaces equals the product of the dimensions of each vector space. The structure maps of the product generator map an element (y_α, y_β) to the vector $i(y_\alpha) \otimes i(y_\beta)$, and a vector x to the function $(d_\alpha \otimes d_\beta)(x)$, where

$$d_\alpha \otimes d_\beta = \overline{d_\alpha \times d_\beta} : \mathbb{X}_\alpha \otimes \mathbb{X}_\beta \to (\mathcal{V}_\mathbb{K}(Y_\alpha), \mu_{Y_\alpha}) \otimes (\mathcal{V}_\mathbb{K}(Y_\beta), \mu_{Y_\beta}) \simeq (\mathcal{V}_\mathbb{K}(Y_\alpha \times Y_\beta), \mu_{Y_\alpha \otimes Y_\beta})$$

is the unique linear extension of the bilinear map defined by

$$(d_\alpha \times d_\beta)(x_\alpha, x_\beta)(y_\alpha, y_\beta) := d_\alpha(x_\alpha)(y_\alpha) \cdot d_\beta(x_\beta)(y_\beta).$$

---

[3] (A1) For any two algebras $\mathbb{X}_\alpha = (X_\alpha, h_\alpha)$ and $\mathbb{X}_\beta = (X_\beta, h_\beta)$ the coequaliser $q_{\mathbb{X}_\alpha,\mathbb{X}_\beta}$ of the algebra homomorphisms $T(h_\alpha \otimes h_\beta)$ and $\mu_{X_\alpha \otimes X_\beta} \circ T(T_{X_\alpha,X_\beta})$ exists (let its codomain be $\mathbb{X}_\alpha \boxtimes \mathbb{X}_\beta := (X_\alpha \boxtimes X_\beta, h_{\alpha \boxtimes \beta})$). (A2) Left and right-tensoring with the induced functor $\boxtimes$ preserves reflexive coequaliser.

$$A = L_{\alpha'\alpha'} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad L_{\alpha\alpha} = \begin{pmatrix} 3 & 2 \\ -5 & -3 \end{pmatrix}, \quad P = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

Figure 6 The basis representation of the counter-clockwise rotation by 90 degree $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L(v) = Av$ with respect to $\alpha = \{(1,2),(1,1)\}$ and $\alpha' = \{(1,0),(0,1)\}$ satisfies $L_{\alpha'\alpha'} = P^{-1}L_{\alpha\alpha}P$.

4.3 Representation Theory

In this section we use our category-theoretical definition of a basis to derive a representation theory for homomorphisms between algebras over monads that is analogous to the representation theory for linear transformations between vector spaces.

In more detail, recall that a linear transformation $L : V \to W$ between k-vector spaces with finite bases $\alpha = \{v_1, ..., v_n\}$ and $\beta = \{w_1, ..., w_m\}$, respectively, admits a matrix representation $L_{\alpha\beta} \in \mathrm{Mat}_k(m,n)$ with $L(v_j) = \sum_i (L_{\alpha\beta})_{i,j} w_i$, such that for any vector v in V the coordinate vectors $L(v)_\beta \in k^m$ and $v_\alpha \in k^n$ satisfy the equality $L(v)_\beta = L_{\alpha\beta} v_\alpha$. A great amount of linear algebra is concerned with finding bases such that the corresponding matrix representation is in an efficient shape, for instance diagonalised. The following definitions generalise the situation by substituting Kleisli morphisms for matrices.

▶ **Definition 24.** *Let $\alpha = (Y_\alpha, i_\alpha, d_\alpha)$ and $\beta = (Y_\beta, i_\beta, d_\beta)$ be bases for T-algebras $\mathbb{X}_\alpha = (X_\alpha, h_\alpha)$ and $\mathbb{X}_\beta = (X_\beta, h_\beta)$, respectively. The basis representation $f_{\alpha\beta}$ of a T-algebra homomorphism $f : \mathbb{X}_\alpha \to \mathbb{X}_\beta$ with respect to α and β is defined by*

$$f_{\alpha\beta} := Y_\alpha \xrightarrow{i_\alpha} X_\alpha \xrightarrow{f} X_\beta \xrightarrow{d_\beta} TY_\beta. \tag{3}$$

Conversely, the morphism $p^{\alpha\beta}$ associated with a Kleisli morphism $p : Y_\alpha \to TY_\beta$ with respect to α and β is defined by

$$p^{\alpha\beta} := X_\alpha \xrightarrow{d_\alpha} TY_\alpha \xrightarrow{Tp} T^2Y_\beta \xrightarrow{\mu_{Y_\beta}} TY_\beta \xrightarrow{Ti_\beta} TX_\beta \xrightarrow{h_\beta} X_\beta. \tag{4}$$

The morphism associated with a Kleisli morphism should be understood as the linear transformation between vector spaces induced by some matrix of the right type. The following result confirms this intuition.

▶ **Lemma 25.** *The function* (4) *is a T-algebra homomorphism $p^{\alpha\beta} : \mathbb{X}_\alpha \to \mathbb{X}_\beta$.*

The next result establishes a generalisation of the observation that for fixed bases, constructing a matrix representation of a linear transformation on the one hand, and associating a linear transformation to a matrix of the right type on the other hand, are mutually inverse operations.

▶ **Lemma 26.** *The operations* (3) *and* (4) *are mutually inverse.*

At the beginning of this section we recalled the soundness identity $L(v)_\beta = L_{\alpha\beta} v_\alpha$ for the matrix representation $L_{\alpha\beta}$ of a linear transformation L. The next result is a natural generalisation of this statement.

▶ **Lemma 27.** *$f_{\alpha\beta}$ is the unique Kleisli-morphism such that $f_{\alpha\beta} \cdot d_\alpha = d_\beta \circ f$. Conversely, $p^{\alpha\beta}$ is the unique T-algebra homomorphism such that $p \cdot d_\alpha = d_\beta \circ p^{\alpha\beta}$.*

The next result establishes the compositionality of the operations (3) and (4). For example, the matrix representation of the composition of two linear transformations is given by the multiplication of the matrix representations of the individual linear transformations.

▶ **Lemma 28.** $g_{\beta\gamma} \cdot f_{\alpha\beta} = (g \circ f)_{\alpha\gamma}$ *and* $q^{\beta\gamma} \circ p^{\alpha\beta} = (q \cdot p)^{\alpha\gamma}$.

The previous statements may be summarised as functors between appropriately defined[4] categories $\mathrm{Alg}_{\mathrm{B}}(T)$ and $\mathrm{Kl}_{\mathrm{B}}(T)$.

▶ **Corollary 29.** *There exist isomorphisms of categories* $\mathrm{BAlg}(T) \simeq \mathrm{Alg}_{\mathrm{B}}(T) \simeq \mathrm{Kl}_{\mathrm{B}}(T)$.

Assume we are given bases α, α' and β, β' for T-algebras (X_α, h_α) and (X_β, h_β), respectively. The following result clarifies how the representations $f_{\alpha\beta}$ and $f_{\alpha'\beta'}$ are related.

▶ **Proposition 30.** *There exist Kleisli isomorphisms p and q such that* $f_{\alpha'\beta'} = q \cdot f_{\alpha\beta} \cdot p$.

Above result simplifies if one restricts to an endomorphism: the basis representations are *similar*. This generalises the situation for vector spaces, cf. Figure 6.

▶ **Proposition 31.** *There exists a Kleisli isomorphism p with Kleisli inverse p^{-1} such that* $f_{\alpha'\alpha'} = p^{-1} \cdot f_{\alpha\alpha} \cdot p$.

4.4 Bases for Bialgebras

This section is concerned with generators and bases for *bialgebras*. It is well-known [39] that an Eilenberg-Moore law λ between a monad T and an endofunctor F induces simultaneously i) a monad $T_\lambda = (T_\lambda, \mu, \eta)$ on $\mathrm{Coalg}(F)$ by $T_\lambda(X, k) = (TX, \lambda_X \circ Tk)$ and $T_\lambda f = Tf$; and ii) an endofunctor F_λ on $\mathrm{Alg}(T)$ by $F_\lambda(X, h) = (FX, Fh \circ \lambda_X)$ and $F_\lambda f = Ff$, such that the algebras over T_λ, the coalgebras of F_λ, and λ-bialgebras coincide. We will consider generators and bases for T_λ-algebras, or equivalently, λ-bialgebras.

By definition, a generator for a λ-bialgebra (X, h, k) consists of a F-coalgebra (Y, k_Y) and morphisms $i : Y \to X$, $d : X \to TY$, such that the three squares on the left of (5) commute:

$$\begin{tikzcd}
Y \arrow[r, "i"] \arrow[d, "k_Y"'] & X \arrow[d, "k"] & X \arrow[r, "d"] \arrow[d, "k"'] & TY \arrow[d, "\lambda_Y \circ Tk_Y"] & TY \arrow[r, "Ti"] & TX \arrow[d, "h"] & TX \arrow[r, "h"] & X \arrow[d, "d"] \\
FY \arrow[r, "Fi"] & FX & FX \arrow[r, "Fd"] & FTY & X \arrow[u, "d"] \arrow[r, "\mathrm{id}_X"] & X & TY \arrow[u, "Ti"] \arrow[r, "\mathrm{id}_{TY}"] & TY
\end{tikzcd} \quad . \tag{5}$$

A basis for a bialgebra is a generator such that the diagram on the right of (5) commutes.

By forgetting the F-coalgebra structure, every generator for a bialgebra is in particular a generator for its underlying T-algebra. By Proposition 6 there exists a λ-bialgebra homomorphism $i^\sharp := h \circ Ti : \exp_T(Y, Fd \circ k \circ i) \to (X, h, k)$. The next result establishes that there exists a second equivalent free bialgebra with a different coalgebra structure.

▶ **Lemma 32.** *Let (Y, k_Y, i, d) be a generator for (X, h, k). Then $i^\sharp : TY \to X$ is a λ-bialgebra homomorphism* $i^\sharp : \mathrm{free}_T(Y, k_Y) \to (X, h, k)$.

If one moves from generators to bases for bialgebras, both structures coincide.

▶ **Lemma 33.** *Let (Y, k_Y, i, d) be a basis for (X, h, k), then* $\mathrm{free}_T(Y, k_Y) = \exp_T(Y, Fd \circ k \circ i)$.

▶ **Example 34** (Canonical RFSA)**.** Recall the minimal pointed bialgebra (X, h, k) for the language $L = (a+b)^*a$ depicted in Figure 2b. Let $(J(\mathbb{X}), i, d)$ be the generator for $\mathbb{X} = (X, h)$ defined as follows: the carrier $J(\mathbb{X})$ consists of the join-irreducibles for $\mathbb{X}$, the embedding

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[4] Let $\mathrm{Alg}_{\mathrm{B}}(T)$ be the category in which objects are given by pairs $(\mathbb{X}_\alpha, \alpha)$, where $\mathbb{X}_\alpha$ is a T-algebra with basis $\alpha = (Y_\alpha, i_\alpha, d_\alpha)$, and a morphism $f : (\mathbb{X}_\alpha, \alpha) \to (\mathbb{X}_\beta, \beta)$ consists of a T-algebra homomorphism $f : \mathbb{X}_\alpha \to \mathbb{X}_\beta$. Let $\mathrm{Kl}_{\mathrm{B}}(T)$ be the category in which objects are the same ones as for $\mathrm{Alg}_{\mathrm{B}}(T)$, and a morphism $p : (\mathbb{X}_\alpha, \alpha) \to (\mathbb{X}_\beta, \beta)$ consists of a Kleisli-morphism $p : Y_\alpha \to TY_\beta$.

satisfies $i(y) = y$, and the decomposition is given by $d(x) = \{y \in J(\mathbb{X}) \mid y \leq x\}$). We used $(J(\mathbb{X}), i, d)$ to recover the canonical RFSA for L depicted in Figure 2c as the coalgebra $(J(\mathbb{X}), Fd \circ k \circ i)$. Examining the graphs shows that k restricts to the join-irreducibles $J(\mathbb{X})$, suggesting $\alpha = (J(\mathbb{X}), k, i, d)$ as a possible generator for the full bialgebra. However, the a-action on $[\{y\}]$ implies the non-commutativity of the second diagram on the left of (5). The issue can be fixed by modifying d via $d([\{y\}]) := \{[\{y\}]\}$. In consequence free$(J(\mathbb{X})), k)$ and $\exp(J(\mathbb{X}), Fd \circ k \circ i)$ coincide (even though the modification does not yield a basis).

We close this section by observing that a basis for the underlying algebra of a bialgebra is sufficient for constructing a generator for the full bialgebra.

▶ **Lemma 35.** *Let (X, h, k) be a λ-bialgebra and (Y, i, d) a basis for the T-algebra (X, h). Then $(TY, (Fd \circ k \circ i)^\sharp, i^\sharp, \eta_{TY} \circ d)$ is a generator for (X, h, k).*

4.5 Bases as Coalgebras

In this section, we compare our approach to an alternative perspective on the generalisation of bases. More specifically, we are interested in the work of Jacobs [18], where a basis is defined as a coalgebra for the comonad on the category of Eilenberg-Moore algebras induced by the free algebra adjunction. Explicitly, a basis for a T-algebra (X, h), in the sense of [18], consists of a T-coalgebra (X, k) such that the following three diagrams commute:

$$\begin{tikzcd} TX \arrow[r, "Tk"] \arrow[d, "h"'] & T^2X \arrow[d, "\mu_X"] \\ X \arrow[r, "k"] & TX \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "k"] \arrow[dr, "\mathrm{id}_X"'] & TX \arrow[d, "h"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "k"] \arrow[d, "k"'] & TX \arrow[d, "T\eta_X"] \\ TX \arrow[r, "Tk"] & T^2X \end{tikzcd} . \tag{6}$$

The next result shows that a basis as in Definition 4 induces a basis in the sense of [18].

▶ **Lemma 36.** *Let (Y, i, d) be a basis for a T-algebra (X, h), then (6) commutes for $k := Ti \circ d$.*

Conversely, assume (X, k) is a T-coalgebra structure satisfying (6) and $i_k : Y_k \to X$ an equaliser of k and η_X. If the underlying category is the category of sets and functions, and Y_k non-empty, one can show that the equaliser is preserved under T, that is, Ti_k is an equaliser of Tk and $T\eta_X$ [18]. Since it holds $Tk \circ k = T\eta_X \circ k$ by (6), there thus exists a unique morphism $d_k : X \to TY_k$, which can be shown to be the inverse of $h \circ Ti_k$ [18]. In other words, $G(X, k) := (Y_k, i_k, d_k)$ is a basis for (X, h) in the sense of Definition 4. In the following let $F(Y, i, d) := (X, Ti \circ d)$ for an arbitrary basis of (X, h).

▶ **Lemma 37.** *Let (Y, i, d) be a basis for a T-algebra (X, h) and $k := Ti \circ d$. Then $\eta_X \circ i = k \circ i$ and $Tk \circ (\eta_X \circ i) = T\eta_X \circ (\eta_X \circ i)$.*

▶ **Corollary 38.** *Let $\alpha := (Y, i, d)$ be a non-empty basis for a set-based T-algebra (X, h) and $k := Ti \circ d$. Then $(\mathrm{id}_{(X,h)})_{\alpha, G F \alpha} : Y \to TY_k$ is the unique morphism making the diagram below commute:*

$$\begin{tikzcd} Y \arrow[r, dashed] \arrow[rr, bend left, "\eta_X \circ i"] & TY_k \arrow[r, "Ti_k"'] & TX \arrow[r, shift left, "Tk"] \arrow[r, shift right, "T\eta_X"'] & T^2X \end{tikzcd} .$$

4.6 Signatures, Equations, and Finitary Monads

Most of the algebras over set monads one usually considers generators for constitute finitary varieties in the sense of universal algebra, i.e. classes of algebraic structures for a finitary signature satisfying a finite set of equations. In this section, we will briefly explore the consequences for generators that arise from this observation.

Formally, a finitary signature consists of a finite set Σ, whose elements are thought of as operations, and a function $\mathrm{ar} : \Sigma \to \mathbb{N}$ that assigns to an operation its finite arity. Any signature induces a set endofunctor H_Σ defined on a set as the coproduct $H_\Sigma X = \coprod_{\sigma \in \Sigma} X^{\mathrm{ar}(\sigma)}$, and consequently, a set monad $\mathbb{S}_\Sigma$ that assigns to a set V of variables the initial algebra $S_\Sigma V = \mu X.V + H_\Sigma X$, i.e. the set of Σ-terms generated by V [38]. One can show that the categories of H_Σ-algebras and $\mathbb{S}_\Sigma$-algebras are isomorphic. We say that a $\mathbb{S}_\Sigma$-algebra $\mathbb{X}$ satisfies a finite set of equations $E \subseteq S_\Sigma V \times S_\Sigma V$, if for all $(s, t) \in E$ and valuations $v : V \to X$ it holds $v^\sharp(s) = v^\sharp(t)$, where $v^\sharp : (S_\Sigma V, \mu_V) \to \mathbb{X}$ is the unique extension of v to a $\mathbb{S}_\Sigma$-algebra homomorphism [4]. The set of $\mathbb{S}_\Sigma$-algebras that satisfy E is denoted by $\mathrm{Alg}(\Sigma, E)$. As one verifies, the forgetful functor $U : \mathrm{Alg}(\Sigma, E) \to \mathrm{Set}$ admits a left-adjoint $F : \mathrm{Set} \to \mathrm{Alg}(\Sigma, E)$, thus resulting in a set monad $T_{\Sigma,E}$ with underlying endofunctor $U \circ F$ that preserves directed colimits. It further can be shown to be monadic, that is, the comparison functor $K : \mathrm{Alg}(\Sigma, E) \to \mathrm{Alg}(T_{\Sigma,E})$ is an isomorphism [23]. In other words, the category of algebras over $T_{\Sigma,E}$ and the finitary variety of algebras over Σ and E coincide. In fact, set monads preserving directed colimits and finitary varieties are in *bijection*. In consequence, monads of such a form are sometimes called *finitary* [4].

The following result characterises generators for algebras over $T_{\Sigma,E}$. It can be seen as a unifying proof for observations analogous to the one in Example 5.

▶ **Lemma 39.** *A morphism $i : Y \to X$ is part of a generator for a $T_{\Sigma,E}$-algebra $\mathbb{X}$ if(f) every element of X can be expressed as a Σ-term in $i[Y]$ modulo E.*

4.7 Finitely Generated Objects

In this section, we relate our abstract definition of a generator to the theory of *locally finitely presentable* categories, in particular, to the notions of *finitely generated* and *finitely presentable* objects, which are categorical abstractions of finitely generated algebraic structures.

For intuition, recall that an element $x \in X$ of a partially ordered set is *finite*, if for each directed set $D \subseteq X$ with $x \leq \bigvee D$, there exists some $d \in D$ satisfying $x \leq d$. An *algebraic lattice* is a partially ordered set that has all joins, and every element is a join of elements that all are finite. The naive categorification of finite elements is equivalent to the following definition: a object Y in $\mathcal{C}$ is *finitely presentable (generated)*, if $\mathrm{Hom}_{\mathcal{C}}(Y, -) : \mathcal{C} \to \mathrm{Set}$ preserves filtered colimits (of monomorphisms). Consequently, one can categorify algebraic lattices as *locally finitely presentable* (lfp) categories, which are cocomplete and admit a set of finitely presentable objects, such that every object is a filtered colimit [4].

For finitary monads on lfp categories, an algebra is a finitely generated object if(f) it is a strong quotient of an algebra generated by a finitely presentable object:

▶ **Lemma 40** ([3])**.** *An algebra $\mathbb{X}$ over a finitary monad T on an lfp category $\mathcal{C}$ is a finitely generated object of* $\mathrm{Alg}(T)$ *if(f) there exists a finitely presentable object Y of $\mathcal{C}$ and a morphism $i : Y \to X$, such that $i^\sharp : TY \to X$ is a strong epimorphism.*

A generator in the sense of Definition 4 requires the existence and choice of a right-inverse to $i^\sharp$, turning the latter into a *split* epimorphism, while above we are only given a *strong* epimorphism. In general, splitness is a relatively heavy quality, as asking for all epimorphisms of some category to be split is equivalent to asserting its internal axiom of choice.

5 Related Work

A central motivation for this paper has been our broad interest in active learning algorithms for state-based models [5]. One of the challenges in learning non-deterministic models is the common lack of a unique minimal acceptor for a given language [16]. The problem has been independently approached for different variants of non-determinism, often with the common idea of finding a subclass admitting a unique representative [17, 10]. Unifying perspectives were given by van Heerdt [42, 40, 41] and Myers et al. [28]. One of the central notions in the work of van Heerdt is the concept of a scoop, originally introduced by Arbib and Manes [6].

In [45] we presented a categorical framework that recovers minimal non-deterministic representatives in two steps. The framework is based on ideas closely related to the ones in [28], adopts scoops under the name generators, and strengthens the former to the notion of a basis. In a first step, it constructs a minimal bialgebra by closing a minimal coalgebra with additional algebraic structure over a monad. In a second step, it identifies generators for the algebraic part of the bialgebra, to derive an equivalent coalgebra with side effects in a monad. In this paper, we generalise the first step as application of a monad on an appropriate category of subobjects with respect to an epi-mono factorisation system, and explore the second step by further developing the abstract theory of generators and bases.

Categorical factorisation systems are well-established [13, 31, 24]. Among others, they have been used for a general view on the minimisation and determinisation of state-based systems [2, 1, 44]. In Section 3 we adapt the formalism of [2]. In Section 3.1 we show that under certain assumptions factorisation systems can be lifted to the categories of algebras and coalgebras. We later realised that the constructions had recently been published in [44].

The notion of a basis for an algebra over an arbitrary monad has been subject of previous interest. Jacobs, for instance, defines a basis as a coalgebra for the comonad on the category of algebras induced by the free algebra adjunction [18]. In Section 4.5 we show that a basis in our sense always induces a basis in their sense, and, conversely, it is possible to recover a basis in our sense from a basis in their sense, if certain assumptions about the existence and preservation of equaliser are given. As equaliser do not necessarily exist and are not necessarily preserved, our approach carries additional data and thus can be seen as finer.

6 Discussion and Future Work

We have generalised the closure of a subset of an algebraic structure as a monad between categories of subobjects relative to a factorisation system. We have identified the closure of a minimal coalgebra as an instance of the closure of subobjects that arise by taking the image of a morphism. We have extended the notion of a generator to a category of algebras with generators, and explored its characteristics. We have generalised the representation theory of vector spaces and discussed bases for bialgebras. We compared our ideas with a coalgebraic generalisation of bases, explored the case in which a monad is induced by a variety, and briefly related our notion to finitely generated objects in finitely presentable categories.

In [45] we have shown that generators and bases in our sense are central ingredients in the definitions of minimal canonical acceptors. Many such acceptors admit double-reversal characterisations [14, 15, 28, 43]. Duality based characterisations as the former have been shown to be closely related to minimisation procedures with respect to factorisation systems [12, 11, 44]. In the future, it would be interesting to further explore the connection between the minimality of generators on the one side, and the minimality of an acceptor with respect to a factorisation system on the other side.

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A Proofs

▶ **Lemma 9** ([44, Lem. 3.6]). *Whenever $g \circ e = m \circ f$ for T-algebra homomorphisms f, g, e, m, with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there exists a unique diagonal T-algebra homomorphism d, such that $f = d \circ e$ and $g = m \circ d$.*

Proof. The proof for [44, Lem. 3.6] consists of a corollary to the dual statement for coalgebras [44, Lem. 3.3]. Below we offer an explicit version for algebras.

We fix the following notation:

$$\begin{tikzcd}
(A, h_A) \arrow[r, two heads, "e"] \arrow[d, "f"'] & (B, h_B) \arrow[d, "g"] \\
(C, h_C) \arrow[r, hook, "m"'] & (D, h_D)
\end{tikzcd}$$

Since any commuting diagram of algebra homomorphism projects to a commuting diagram in $\mathcal{C}$, the factorisation system of $\mathcal{C}$ implies the existence of a unique diagonal d in $\mathcal{C}$. It remains to show that d is an algebra homomorphism, that is, we need to establish the following identity:

$$h_C \circ Td = d \circ h_B.$$

To this end, we observe that since the following two diagrams commute

$$\begin{tikzcd}
TA \arrow[rrr, "Te"] \arrow[d, "h_A"'] \arrow[dr, "Tf"] & & & TB \arrow[d, "h_B"] \arrow[dll, "Td"'] \arrow[dl, "Tg"] \\
A \arrow[d, "f"'] & TC \arrow[r, "Tm"'] \arrow[dl, "h_C"] & TD \arrow[dr, "h_D"] & B \arrow[d, "g"] \\
C \arrow[rrr, "m"'] & & & D
\end{tikzcd}
\qquad
\begin{tikzcd}
TA \arrow[rr, "Te"] \arrow[d, "h_A"'] & & TB \arrow[d, "h_B"] \arrow[dl, "h_B"] \\
A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[dl, "d"] \arrow[dr, "g"] & B \arrow[d, "g"] \\
C \arrow[rr, "m"'] & & D
\end{tikzcd}$$

both $h_C \circ Td$ and $d \circ h_B$ are solutions to the unique diagonal below:

$$\begin{tikzcd}
TA \arrow[r, two heads, "Te"] \arrow[d, "f \circ h_A"'] & TB \arrow[d, "g \circ h_B"] \arrow[dl, dashed] \\
C \arrow[r, hook, "m"'] & D
\end{tikzcd} .$$

◀

▶ **Lemma 10** ([44, Prop. 3.7]). *$(\mathrm{im}(f), h_{\mathrm{im}(f)})$ is an Eilenberg-Moore T-algebra.*

Proof. We need to establish the following two identities

$$h_{\mathrm{im}(f)} \circ \eta_{\mathrm{im}(f)} = \mathrm{id}_{\mathrm{im}(f)}$$
$$h_{\mathrm{im}(f)} \circ \mu_{\mathrm{im}(f)} = h_{\mathrm{im}(f)} \circ Th_{\mathrm{im}(f)}$$

where the latter captures that $h_{\mathrm{im}(f)} : (T\mathrm{im}(f), \mu_{\mathrm{im}(f)}) \to (\mathrm{im}(f), h_{\mathrm{im}(f)})$ is a T-algebra homomorphism.

For the first equality, we observe (as in the proof for ([44, Prop. 3.7])) that since the diagram below commutes

$$\begin{tikzcd}
X \arrow[r, "1"] \arrow[dd, "e"'] & X \arrow[rrr, "e"] \arrow[d, "\eta_X"'] & & & \mathrm{im}(f) \arrow[d, "m"] \arrow[dll, "\eta_{\mathrm{im}(f)}"] \\
& TX \arrow[r, "Te"] \arrow[d, "e \circ h_X"'] & T\mathrm{im}(f) \arrow[r, "Tm"] \arrow[dl, "h_{\mathrm{im}(f)}"] & TY \arrow[dr, "h_Y"] & Y \arrow[l, "\eta_Y"'] \arrow[d, "1"] \\
\mathrm{im}(f) \arrow[r, "1"'] & \mathrm{im}(f) \arrow[rrr, "m"'] & & & Y
\end{tikzcd}$$

both $h_{\mathrm{im}(f)} \circ \eta_{\mathrm{im}(f)}$ and $\mathrm{id}_{\mathrm{im}(f)}$ are solutions to the unique diagonal in $\mathcal{C}$ below:

$$\begin{tikzcd}
X \arrow[r, two heads, "e"] \arrow[d, "e"'] & \mathrm{im}(f) \arrow[d, "m"] \arrow[dl, dashed] \\
\mathrm{im}(f) \arrow[r, hook, "m"'] & Y
\end{tikzcd} .$$

Similarly, for the second equality, we observe that since the following two diagrams commute

$$\begin{tikzcd}
T^2X \arrow[rrr, "T^2e"] \arrow[dd, "\mu_X"'] & & & T^2\mathrm{im}(f) \arrow[d, "T^2m"] \arrow[ddll, "\mu_{\mathrm{im}(f)}"] \\
& & & T^2Y \arrow[d, "Th_Y"] \arrow[dl, "\mu_Y"] \\
TX \arrow[r, "Te"] \arrow[d, "e \circ h_X"'] & T\mathrm{im}(f) \arrow[r, "Tm"] \arrow[d, "h_{\mathrm{im}(f)}"] & TY \arrow[dr, "h_Y"] & TY \arrow[d, "h_Y"] \\
\mathrm{im}(f) \arrow[r, "1"'] & \mathrm{im}(f) \arrow[rr, "m"'] & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2X \arrow[r, "1"] \arrow[d, "\mu_X"'] & T^2X \arrow[r, "T^2e"] \arrow[d, "Th_X"] & T^2\mathrm{im}(f) \arrow[d, "T^2m"] \arrow[ddl, "Th_{\mathrm{im}(f)}"] \\
TX \arrow[d, "h_X"'] & TX \arrow[d, "Te"] \arrow[dl, "h_X"] & T^2Y \arrow[d, "Th_Y"] \\
X \arrow[d, "e"'] & T\mathrm{im}(f) \arrow[r, "Tm"] \arrow[dl, "h_{\mathrm{im}(f)}"] & TY \arrow[d, "h_Y"] \\
\mathrm{im}(f) \arrow[rr, "m"'] & & Y
\end{tikzcd}$$

both $h_{\mathrm{im}(f)} \circ \mu_{\mathrm{im}(f)}$ and $h_{\mathrm{im}(f)} \circ Th_{\mathrm{im}(f)}$ are solutions to the unique diagonal below:

$$\begin{tikzcd}
T^2X \arrow[r, two heads, "T^2e"] \arrow[d, "e \circ h_X \circ \mu_X"'] & T^2\mathrm{im}(f) \arrow[d, "h_Y \circ Th_Y \circ T^2m"] \arrow[dl, dashed] \\
\mathrm{im}(f) \arrow[r, hook, "m"'] & Y
\end{tikzcd}$$

Alternatively (as in the proof for [44, Prop. 3.7]), one may observe that since the following outer square of homomorphisms between algebras for the endofunctor T commutes

$$\begin{tikzcd}
(TX, \mu_X) \arrow[r, two heads, "Te"] \arrow[d, "e \circ h_X"'] & (T\mathrm{im}(f), \mu_{\mathrm{im}(f)}) \arrow[d, "h_Y \circ Tm"] \arrow[dl, dashed] \\
(\mathrm{im}(f), h_{\mathrm{im}(f)}) \arrow[r, hook, "m"'] & (Y, h_Y)
\end{tikzcd}$$

Lemma 9 implies the existence of a unique diagonal algebra homomorphism making the two triangles above commute. As we know that the diagonal coincides with the unique diagonal of the corresponding diagram in $\mathcal{C}$, which is given by $h_{\mathrm{im}(f)}$, we can deduce that the latter is a T-algebra homomorphism. ◀

▶ **Lemma 41.** *For any morphism $f : m_{Y_1} \to m_{Y_2}$ between subobjects of X, the following diagram commutes:*

$$\begin{tikzcd}
(TY_1, \mu_{Y_1}) \arrow[r, two heads, "e_{\overline{Y_1}}"] \arrow[d, "e_{\overline{Y_2}} \circ Tf"'] & (\overline{Y_1}, h_{\overline{Y_1}}) \arrow[d, "m_{\overline{Y_1}}"] \\
(\overline{Y_2}, h_{\overline{Y_2}}) \arrow[r, hook, "m_{\overline{Y_2}}"'] & (X, h)
\end{tikzcd} .$$

Proof. The statement follows from the commutativity of the inner diagrams below:

$$\begin{tikzcd}
(TY_1, \mu_{Y_1}) \arrow[rr, two heads, "e_{\overline{Y_1}}"] \arrow[d, "Tf"'] \arrow[dr, "Tm_{Y_1}"] & & (\overline{Y_1}, h_{\overline{Y_1}}) \arrow[dd, "m_{\overline{Y_1}}"] \\
(TY_2, \mu_{Y_2}) \arrow[r, "Tm_{Y_2}"'] \arrow[d, "e_{\overline{Y_2}}"'] & (TX, \mu_X) \arrow[dr, "h"] & \\
(\overline{Y_2}, h_{\overline{Y_2}}) \arrow[rr, hook, "m_{\overline{Y_2}}"'] & & (X, h)
\end{tikzcd} .$$

◀

▶ **Proposition 11.** *Assigning $m_Y \mapsto m_{\overline{Y}}$ and $f \mapsto \overline{f}$ yields a functor $\overline{(\cdot)}^{\mathbb{X}} : \mathrm{Sub}(X) \to \mathrm{Sub}(\mathbb{X})$.*

Proof. We need to establish the identities

$$\overline{\mathrm{id}_Y} = \mathrm{id}_{\overline{Y}} \quad \text{and} \quad \overline{f \circ g} = \overline{f} \circ \overline{g}.$$

As before, the equations follow from the uniqueness of diagonals and the commutativity of the left, respectively right, diagram below:

$$\begin{tikzcd}
(TY, \mu_Y) \arrow[r, two heads, "e_{\overline{Y}}"] \arrow[d, "e_{\overline{Y}} \circ T(\mathrm{id}_Y)"'] & (\overline{Y}, h_{\overline{Y}}) \arrow[dl, "\mathrm{id}_{\overline{Y}}"] \arrow[d, "m_{\overline{Y}}"] \\
(\overline{Y}, h_{\overline{Y}}) \arrow[r, hook, "m_{\overline{Y}}"'] & (X, h)
\end{tikzcd}
\qquad
\begin{tikzcd}
(TY_1, \mu_{Y_1}) \arrow[rr, two heads, "e_{\overline{Y_1}}"] \arrow[d, "Tg"'] & & (\overline{Y_1}, h_{\overline{Y_1}}) \arrow[dl, "\overline{g}"] \arrow[ddd, "m_{\overline{Y_1}}"] \\
(TY_2, \mu_{Y_2}) \arrow[r, "e_{\overline{Y_2}}"] \arrow[d, "Tf"'] & (\overline{Y_2}, h_{\overline{Y_2}}) \arrow[ddl, "\overline{f}"] \arrow[ddr, "m_{\overline{Y_2}}"] & \\
(TY_3, \mu_{Y_3}) \arrow[d, "e_{\overline{Y_3}}"'] & & \\
(\overline{Y_3}, h_{\overline{Y_3}}) \arrow[rr, hook, "m_{\overline{Y_3}}"'] & & (X, h)
\end{tikzcd} .$$

◀

▶ **Lemma 12.** *The following two equalities hold:*

- $m_{\overline{Y}} \circ e_{\overline{Y}} \circ \eta_Y = m_Y$
- $m_{\overline{Y}} \circ e_{\overline{Y}} \circ \mu_Y = m_{\overline{\overline{Y}}} \circ e_{\overline{\overline{Y}}} \circ Te_{\overline{Y}}$

Proof. For the first identity we observe:

$$\begin{aligned}
m_{\overline{Y}} \circ e_{\overline{Y}} \circ \eta_Y &= m^\sharp_Y \circ \eta_Y && (m_{\overline{Y}} \circ e_{\overline{Y}} = m^\sharp_Y)\\
&= h \circ Tm_Y \circ \eta_Y && (\text{Definition of } m^\sharp_Y)\\
&= h \circ \eta_X \circ m_Y && (\text{Naturality of } \eta)\\
&= m_Y && (h \circ \eta_X = \mathrm{id}_X).
\end{aligned}$$

Similarly, for the second identity we deduce:

$$\begin{aligned}
m_{\overline{Y}} \circ e_{\overline{Y}} \circ \mu_Y &= m^\sharp_Y \circ \mu_Y && (m_{\overline{Y}} \circ e_{\overline{Y}} = m^\sharp_Y)\\
&= h \circ Tm_Y \circ \mu_Y && (\text{Definition of } m^\sharp_Y)\\
&= h \circ \mu_X \circ T^2 m_Y && (\text{Naturality of } \mu)\\
&= h \circ Th \circ T^2 m_Y && (h \circ \mu_X = h \circ Th)\\
&= h \circ Tm^\sharp_Y && (\text{Definition of } m^\sharp_Y)\\
&= h \circ Tm_{\overline{Y}} \circ Te_{\overline{Y}} && (m^\sharp_Y = m_{\overline{Y}} \circ e_{\overline{Y}})\\
&= m^\sharp_{\overline{Y}} \circ Te_{\overline{Y}} && (\text{Definition of } m^\sharp_{\overline{Y}})\\
&= m_{\overline{\overline{Y}}} \circ e_{\overline{\overline{Y}}} \circ Te_{\overline{Y}} && (m^\sharp_{\overline{Y}} = m_{\overline{\overline{Y}}} \circ e_{\overline{\overline{Y}}}).
\end{aligned}$$

◀

▶ **Theorem 13.** *$(\overline{(\cdot)}^{\mathbb{X}}, \eta^{\mathbb{X}}, \mu^{\mathbb{X}})$ is a monad on* $\mathrm{Sub}(X)$*.*

Proof. On the one hand, we have to establish the naturality of $\eta^{\mathbb{X}}$ and $\mu^{\mathbb{X}}$, that is, the identities

$$\begin{aligned}
&\overline{f} \circ \eta^{\mathbb{X}}_{m_{Y_1}} = \eta^{\mathbb{X}}_{m_{Y_2}} \circ f\\
&\overline{f} \circ \mu^{\mathbb{X}}_{m_{Y_1}} = \mu^{\mathbb{X}}_{m_{Y_2}} \circ \overline{\overline{f}}
\end{aligned} \tag{7}$$

for any subobject homomorphism $f : m_{Y_1} \to m_{Y_2}$. On the other hand, we need to establish the unitality and associativity laws:

$$\begin{aligned}
&\mu^{\mathbb{X}}_{m_Y} \circ \eta^{\mathbb{X}}_{m_{\overline{Y}}} = \mathrm{id}_{\overline{Y}} = \mu^{\mathbb{X}}_{m_Y} \circ \overline{\eta^{\mathbb{X}}_{m_Y}}\\
&\mu^{\mathbb{X}}_{m_Y} \circ \mu^{\mathbb{X}}_{m_{\overline{Y}}} = \mu^{\mathbb{X}}_{m_Y} \circ \overline{\mu^{\mathbb{X}}_{m_Y}}.
\end{aligned} \tag{8}$$

The first equation of (7) follows from the commutativity of the diagram below:

$$\begin{tikzcd}
Y_1 \arrow[r, "1"] \arrow[dd, swap, "\eta^{\mathbb{X}}_{m_{Y_1}}"] & Y_1 \arrow[r, "f"] \arrow[d, "\eta_{Y_1}"] & Y_2 \arrow[r, "1"] \arrow[d, "\eta_{Y_1}"] & Y_2 \arrow[dd, "\eta^{\mathbb{X}}_{m_{Y_2}}"] \\
 & TY_1 \arrow[r, "Tf"] \arrow[d, "e_{\overline{Y_1}}"] & TY_2 \arrow[dr, "e_{\overline{Y_2}}"] & \\
\overline{Y_1} \arrow[r, swap, "1"] & \overline{Y_1} \arrow[rr, swap, "\overline{f}"] & & \overline{Y_2}
\end{tikzcd} .$$

For the second equation of (7) we observe that since the following two diagrams commute

$$\begin{tikzcd}
T^2Y_1 \arrow[rr, "e_{\overline{\overline{Y_1}}}\circ Te_{\overline{Y_1}}"] \arrow[d, "\mu_{Y_1}"'] & & \overline{\overline{Y_1}} \arrow[dl, "\mu^{\mathbb{X}}_{m_{Y_1}}"'] \arrow[dd, "m_{\overline{\overline{Y_1}}}"] \\
TY_1 \arrow[r, "e_{\overline{Y_1}}"] \arrow[d, "e_{\overline{Y_2}}\circ Tf"'] & \overline{Y_1} \arrow[dl, "\overline{f}"] \arrow[dr, "m_{\overline{Y_1}}"] & \\
\overline{Y_2} \arrow[rr, "m_{\overline{Y_2}}"'] & & X
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2Y_1 \arrow[r, "1"] \arrow[d, "Tf\circ\mu_{Y_1}"'] & T^2Y_1 \arrow[r, "Te_{\overline{Y_1}}"] \arrow[d, "T^2f"] & T\overline{Y_1} \arrow[r, "e_{\overline{\overline{Y_1}}}"] \arrow[d, "T\overline{f}"] & \overline{\overline{Y_1}} \arrow[ddl, "\overline{\overline{f}}"] \arrow[ddd, "m_{\overline{\overline{Y_1}}}"] \\
TY_2 \arrow[ddd, "e_{\overline{Y_2}}"'] & T^2Y_2 \arrow[l, "\mu_{Y_2}"] \arrow[r, "Te_{\overline{Y_2}}"] & T\overline{Y_2} \arrow[d, "e_{\overline{\overline{Y_2}}}"'] & \\
 & & \overline{\overline{Y_2}} \arrow[dll, "\mu^{\mathbb{X}}_{m_{Y_2}}"] \arrow[dr, "m_{\overline{\overline{Y_2}}}"] & \\
\overline{Y_2} \arrow[rrr, "m_{\overline{Y_2}}"'] & & & X
\end{tikzcd}$$

both $\overline{f}\circ\mu^{\mathbb{X}}_{m_{Y_1}}$ and $\mu^{\mathbb{X}}_{m_{Y_2}}\circ\overline{\overline{f}}$ are solutions to the unique diagonal below:

$$\begin{tikzcd}
T^2Y_1 \arrow[r, two heads, "e_{\overline{\overline{Y_1}}}\circ Te_{\overline{Y_1}}"] \arrow[d, "e_{\overline{Y_2}}\circ Tf\circ\mu_{Y_1}"'] & \overline{\overline{Y_1}} \arrow[d, "m_{\overline{\overline{Y_1}}}"] \arrow[dl, dashed] \\
\overline{Y_2} \arrow[r, hook, "m_{\overline{Y_2}}"'] & X
\end{tikzcd} .$$

For the first equation of (8) we observe that since the following diagrams commute

$$\begin{tikzcd}
TY \arrow[r, "1"] \arrow[dd, "1"'] & TY \arrow[r, "e_{\overline{Y}}"] \arrow[d, "\eta_{TY}"'] & \overline{Y} \arrow[r, "1"] \arrow[d, "\eta_{\overline{Y}}"'] & \overline{Y} \arrow[ddl, "\eta^{\mathbb{X}}_{m_{\overline{Y}}}"] \arrow[ddd, "m_{\overline{Y}}"] \\
 & T^2Y \arrow[r, "Te_{\overline{Y}}"] \arrow[d, "\mu_Y"'] & T\overline{Y} \arrow[d, "e_{\overline{\overline{Y}}}"'] & \\
TY \arrow[r, "1"] \arrow[d, "e_{\overline{Y}}"'] & TY \arrow[d, "e_{\overline{Y}}"'] & \overline{\overline{Y}} \arrow[dl, "\mu^{\mathbb{X}}_{m_Y}"] \arrow[dr, "m_{\overline{\overline{Y}}}"] & \\
\overline{Y} \arrow[r, "1"'] & \overline{Y} \arrow[rr, "m_{\overline{Y}}"'] & & X
\end{tikzcd}
\qquad
\begin{tikzcd}
TY \arrow[r, "1"] \arrow[dd, "1"'] & TY \arrow[r, "1"] \arrow[d, "T\eta_Y"'] & TY \arrow[r, "e_{\overline{Y}}"] \arrow[d, "T\eta^{\mathbb{X}}_{m_Y}"'] & \overline{Y} \arrow[ddl, "\overline{\eta^{\mathbb{X}}_{m_Y}}"] \arrow[ddd, "m_{\overline{Y}}"] \\
 & T^2Y \arrow[r, "Te_{\overline{Y}}"] \arrow[d, "\mu_Y"'] & T\overline{Y} \arrow[d, "e_{\overline{\overline{Y}}}"'] & \\
TY \arrow[r, "1"] \arrow[d, "e_{\overline{Y}}"'] & TY \arrow[d, "e_{\overline{Y}}"'] & \overline{\overline{Y}} \arrow[d, "\mu^{\mathbb{X}}_{m_Y}"'] \arrow[dr, "m_{\overline{\overline{Y}}}"] & \\
\overline{Y} \arrow[r, "1"'] & \overline{Y} \arrow[r, "1"'] & \overline{Y} \arrow[r, "m_{\overline{Y}}"'] & X
\end{tikzcd}$$

all three morphisms $\mu^{\mathbb{X}}_{m_Y}\circ\eta^{\mathbb{X}}_{m_{\overline{Y}}}$, $\mathrm{id}_{\overline{Y}}$ and $\mu^{\mathbb{X}}_{m_Y}\circ\overline{\eta^{\mathbb{X}}_{m_Y}}$ are solutions to the unique diagonal:

$$\begin{tikzcd}
TY \arrow[r, two heads, "e_{\overline{Y}}"] \arrow[d, "e_{\overline{Y}}"'] & \overline{Y} \arrow[d, "m_{\overline{Y}}"] \arrow[dl, dashed] \\
\overline{Y} \arrow[r, hook, "m_{\overline{Y}}"'] & X
\end{tikzcd} .$$

Similarly, for the second equation of (8) we note that since the following two diagrams commute

$$\begin{tikzcd}
T^3Y \arrow[r, "T^2e_{\overline{Y}}"] \arrow[d, "\mu_{TY}"'] & T^2\overline{X} \arrow[r, "e_{\overline{\overline{\overline{Y}}}}\circ Te_{\overline{\overline{Y}}}"] \arrow[d, "\mu_{\overline{Y}}"'] & \overline{\overline{\overline{Y}}} \arrow[ddl, "\mu^{\mathbb{X}}_{m_{\overline{Y}}}"] \arrow[ddd, "m_{\overline{\overline{\overline{Y}}}}"] \\
T^2Y \arrow[r, "Te_{\overline{Y}}"] \arrow[dd, "e_{\overline{Y}}\circ\mu_Y"'] & T\overline{Y} \arrow[d, "e_{\overline{\overline{Y}}}"'] & \\
 & \overline{\overline{Y}} \arrow[dl, "\mu^{\mathbb{X}}_{m_Y}"] \arrow[dr, "m_{\overline{\overline{Y}}}"] & \\
\overline{Y} \arrow[rr, "m_{\overline{Y}}"'] & & X
\end{tikzcd}
\qquad
\begin{tikzcd}
T^3Y \arrow[r, "1"] \arrow[d, "\mu_{TY}"'] & T^3Y \arrow[r, "Te_{\overline{\overline{Y}}}\circ T^2e_{\overline{Y}}"] \arrow[d, "T\mu_Y"'] & T\overline{\overline{Y}} \arrow[r, "e_{\overline{\overline{\overline{Y}}}}"] \arrow[d, "T\mu^{\mathbb{X}}_{m_Y}"'] & \overline{\overline{\overline{Y}}} \arrow[ddl, "\overline{\mu^{\mathbb{X}}_{m_Y}}"] \arrow[ddd, "m_{\overline{\overline{\overline{Y}}}}"] \\
T^2Y \arrow[d, "\mu_Y"'] & T^2Y \arrow[r, "Te_{\overline{Y}}"'] \arrow[d, "\mu_Y"'] & T\overline{Y} \arrow[d, "e_{\overline{\overline{Y}}}"'] & \\
TY \arrow[r, "1"] \arrow[d, "e_{\overline{Y}}"'] & TY \arrow[d, "e_{\overline{Y}}"] & \overline{\overline{Y}} \arrow[dl, "\mu^{\mathbb{X}}_{m_Y}"] \arrow[dr, "m_{\overline{\overline{Y}}}"] & \\
\overline{Y} \arrow[r, "1"'] & \overline{Y} \arrow[rr, "m_{\overline{Y}}"'] & & X
\end{tikzcd}$$

both morphisms $\mu^{\mathbb{X}}_{m_Y} \circ \mu^{\mathbb{X}}_{m_{\overline{Y}}}$ and $\mu^{\mathbb{X}}_{m_Y} \circ \overline{\mu^{\mathbb{X}}_{m_Y}}$ are solutions to the unique diagonal below:

$$\begin{tikzcd}
T^3Y \arrow[r, two heads, "e_{\overline{\overline{\overline{Y}}}} \circ Te_{\overline{\overline{Y}}} \circ T^2 e_{\overline{Y}}"] \arrow[d, "e_{\overline{Y}} \circ \mu_Y \circ \mu_{TY}"'] & \overline{\overline{\overline{Y}}} \arrow[d, "m_{\overline{\overline{\overline{Y}}}}"] \arrow[dl, dashed] \\
\overline{Y} \arrow[r, hook, "m_{\overline{Y}}"'] & X
\end{tikzcd} .$$

◀

▶ **Lemma 14.** *For any* $f : \mathbb{A} \to \mathbb{B} \in \mathcal{M}$*, there exists a monad morphism* $(f_*, \alpha) : \overline{(\cdot)}^{\mathbb{A}} \to \overline{(\cdot)}^{\mathbb{B}}$.

Proof. We need to define a natural transformation $\alpha : \overline{(\cdot)}^{\mathbb{B}} \circ f_\star \Rightarrow f_\star \circ \overline{(\cdot)}^{\mathbb{A}}$ between functors of type $\mathrm{Sub}(A) \to \mathrm{Sub}(B)$. That is, for any subobject $m_X : X \to A$, we require a homomorphism

$$\alpha_{m_X} : m_{\overline{X}^{\mathbb{B}}} \to f \circ m_{\overline{X}^{\mathbb{A}}}$$

between subobjects of B. Since factorisations are unique up to unique isomorphism, and the diagram on the left below commutes

$$\begin{tikzcd}
TX \arrow[rrr, two heads, "e_{\overline{X}^{\mathbb{B}}}"] \arrow[dr, "Tm_X"] \arrow[dd, two heads, "e_{\overline{X}^{\mathbb{A}}}"'] & & & \overline{X}^{\mathbb{B}} \arrow[dd, hook, "m_{\overline{X}^{\mathbb{B}}}"] \\
& TA \arrow[r, "Tf"] \arrow[d, "h_A"] & TB \arrow[dr, "h_B"] & \\
\overline{X}^{\mathbb{A}} \arrow[r, hook, "m_{\overline{X}^{\mathbb{A}}}"'] & A \arrow[rr, hook, "f"'] & & B
\end{tikzcd}
\qquad
\begin{tikzcd}
TX \arrow[r, two heads, "e_{\overline{X}^{\mathbb{B}}}"] \arrow[d, two heads, "e_{\overline{X}^{\mathbb{A}}}"'] & \overline{X}^{\mathbb{B}} \arrow[d, hook, "m_{\overline{X}^{\mathbb{B}}}"] \arrow[dl, dashed, "\phi_{m_X}"] \\
\overline{X}^{\mathbb{A}} \arrow[r, hook, "f \circ m_{\overline{X}^{\mathbb{A}}}"'] & B
\end{tikzcd}$$

there exists a unique homomorphism $\phi_{m_X} : m_{\overline{X}^{\mathbb{B}}} \to f \circ m_{\overline{X}^{\mathbb{A}}}$ of subobjects of B as indicated on the right above. We thus propose the definition

$$\alpha_{m_X} := \phi_{m_X}.$$

We begin by showing that above proposal turns α into a *natural* transformation. Let $g : m_X \to m_Y$ be a morphism of subobjects of A and $f_\star(g) = g : f_\star(m_X) \to f_\star(m_Y)$ the induced morphism of subobjects of B. We need to prove the equality

$$\phi_{m_Y} \circ \overline{f_\star(g)}^{\mathbb{B}} = \overline{g}^{\mathbb{A}} \circ \phi_{m_X}.$$

To this end, note that, as the two diagrams below commute

$$\begin{tikzcd}
TX \arrow[rr, "e_{\overline{X}^{\mathbb{B}}}"] \arrow[dr, "e_{\overline{X}^{\mathbb{A}}}"] \arrow[dd, "e_{\overline{Y}^{\mathbb{A}}} \circ Tg"'] & & \overline{X}^{\mathbb{B}} \arrow[dl, "\phi_{m_X}"] \arrow[dd, "m_{\overline{X}^{\mathbb{B}}}"] \\
& \overline{X}^{\mathbb{A}} \arrow[dl, "\overline{g}^{\mathbb{A}}"] \arrow[dr, "f \circ m_{\overline{X}^{\mathbb{A}}}"] & \\
\overline{Y}^{\mathbb{A}} \arrow[rr, "f \circ m_{\overline{Y}^{\mathbb{A}}}"'] & & B
\end{tikzcd}
\qquad
\begin{tikzcd}
TX \arrow[rr, "e_{\overline{X}^{\mathbb{B}}}"] \arrow[d, "Tg"'] & & \overline{X}^{\mathbb{B}} \arrow[dl, "\overline{f_\star(g)}^{\mathbb{B}}"'] \arrow[dd, "m_{\overline{X}^{\mathbb{B}}}"] \\
TY \arrow[r, "e_{\overline{Y}^{\mathbb{B}}}"] \arrow[d, "e_{\overline{Y}^{\mathbb{A}}}"'] & \overline{Y}^{\mathbb{B}} \arrow[dl, "\phi_{m_Y}"] \arrow[dr, "m_{\overline{Y}^{\mathbb{B}}}"] & \\
\overline{Y}^{\mathbb{A}} \arrow[rr, "f \circ m_{\overline{Y}^{\mathbb{A}}}"'] & & B
\end{tikzcd}$$

both $\phi_{m_Y} \circ \overline{f_\star(g)}^{\mathbb{B}}$ and $\overline{g}^{\mathbb{A}} \circ \phi_{m_X}$ are solutions to the unique diagonal below:

$$\begin{tikzcd}
TX \arrow[r, two heads, "e_{\overline{X}^{\mathbb{B}}}"] \arrow[d, "e_{\overline{Y}^{\mathbb{A}}} \circ Tg"'] & \overline{X}^{\mathbb{B}} \arrow[d, "m_{\overline{X}^{\mathbb{B}}}"] \arrow[dl, dashed] \\
\overline{Y}^{\mathbb{A}} \arrow[r, hook, "f \circ m_{\overline{Y}^{\mathbb{A}}}"'] & B
\end{tikzcd} .$$

Finally, one verifies that the commutative diagrams turning $(f_\star, \alpha)$ into a morphism between monads correspond to the two equations

$$\phi_{m_X} \circ \mu^{\mathbb{B}}_{f_\star(m_X)} = \mu^{\mathbb{A}}_{m_X} \circ \phi_{m_{\overline{X}^{\mathbb{A}}}} \circ \overline{\phi_{m_X}}^{\mathbb{B}} \qquad \eta^{\mathbb{A}}_{m_X} = \phi_{m_X} \circ \eta^{\mathbb{B}}_{f_\star(m_X)}.$$

For the first equation we observe that since the two diagrams below commute

$$\begin{tikzcd}
T^2X \arrow[rr, "e_{\overline{\overline{X}^{\mathbb{B}}}^{\mathbb{B}}} \circ Te_{\overline{X}^{\mathbb{B}}}"] \arrow[d, "\mu_X"'] & & \overline{\overline{X}^{\mathbb{B}}}^{\mathbb{B}} \arrow[dl, "\mu^{\mathbb{B}}_{f_\star(m_X)}"'] \arrow[dd, "m_{\overline{\overline{X}^{\mathbb{B}}}^{\mathbb{B}}}"] \\
TX \arrow[r, "e_{\overline{X}^{\mathbb{B}}}"] \arrow[d, "e_{\overline{X}^{\mathbb{A}}}"'] & \overline{X}^{\mathbb{B}} \arrow[dl, "\phi_{m_X}"] \arrow[dr, "m_{\overline{X}^{\mathbb{B}}}"] & \\
\overline{X}^{\mathbb{A}} \arrow[rr, "f \circ m_{\overline{X}^{\mathbb{A}}}"'] & & B
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2X \arrow[r, "Te_{\overline{X}^{\mathbb{B}}}"] \arrow[dr, "Te_{\overline{X}^{\mathbb{A}}}"] \arrow[ddd, "e_{\overline{X}^{\mathbb{A}}} \circ \mu_X"'] & T\overline{X}^{\mathbb{B}} \arrow[rr, "e_{\overline{\overline{X}^{\mathbb{B}}}^{\mathbb{B}}}"] \arrow[d, "T\phi_{m_X}"] & & \overline{\overline{X}^{\mathbb{B}}}^{\mathbb{B}} \arrow[dl, "\overline{\phi_{m_X}}^{\mathbb{B}}"'] \arrow[ddd, "m_{\overline{\overline{X}^{\mathbb{B}}}^{\mathbb{B}}}"] \\
& T\overline{X}^{\mathbb{A}} \arrow[r, "e_{\overline{\overline{X}^{\mathbb{A}}}^{\mathbb{B}}}"] \arrow[d, "e_{\overline{\overline{X}^{\mathbb{A}}}^{\mathbb{A}}}"'] & \overline{\overline{X}^{\mathbb{A}}}^{\mathbb{B}} \arrow[dl, "\phi_{m_{\overline{X}^{\mathbb{A}}}}"] \arrow[ddr, "m_{\overline{\overline{X}^{\mathbb{A}}}^{\mathbb{B}}}"] & \\
& \overline{\overline{X}^{\mathbb{A}}}^{\mathbb{A}} \arrow[dl, "\mu^{\mathbb{A}}_{m_X}"] \arrow[drr, "f \circ m_{\overline{\overline{X}^{\mathbb{A}}}^{\mathbb{A}}}"] & & \\
\overline{X}^{\mathbb{A}} \arrow[rrr, "f \circ m_{\overline{X}^{\mathbb{A}}}"'] & & & B
\end{tikzcd}$$

both $\phi_{m_X} \circ \mu^{\mathbb{B}}_{f_\star(m_X)}$ and $\mu^{\mathbb{A}}_{m_X} \circ \phi_{m_{\overline{X}^{\mathbb{A}}}} \circ \overline{\phi_{m_X}}^{\mathbb{B}}$ are solutions to the unique diagonal below:

$$\begin{tikzcd}
T^2X \arrow[r, two heads, "e_{\overline{\overline{X}^{\mathbb{B}}}^{\mathbb{B}}} \circ Te_{\overline{X}^{\mathbb{B}}}"] \arrow[d, "e_{\overline{X}^{\mathbb{A}}} \circ \mu_X"'] & \overline{\overline{X}^{\mathbb{B}}}^{\mathbb{B}} \arrow[d, "m_{\overline{\overline{X}^{\mathbb{B}}}^{\mathbb{B}}}"] \arrow[dl, dashed] \\
\overline{X}^{\mathbb{A}} \arrow[r, hook, "f \circ m_{\overline{X}^{\mathbb{A}}}"'] & B
\end{tikzcd}.$$

For the second equation we observe that since the two diagrams below commute

$$\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, "e_{\overline{X}^{\mathbb{A}}} \circ \eta_X"'] & X \arrow[dl, "\eta^{\mathbb{A}}_{m_X}"] \arrow[d, "f \circ m_X"] \\
\overline{X}^{\mathbb{A}} \arrow[r, "f \circ m_{\overline{X}^{\mathbb{A}}}"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, "1"] \arrow[d, "\eta_X"'] & & X \arrow[dl, "\eta^{\mathbb{B}}_{f_\star(m_X)}"'] \arrow[dd, "f \circ m_X"] \\
TX \arrow[r, "e_{\overline{X}^{\mathbb{B}}}"] \arrow[d, "e_{\overline{X}^{\mathbb{A}}}"'] & \overline{X}^{\mathbb{B}} \arrow[dl, "\phi_{m_X}"] \arrow[dr, "m_{\overline{X}^{\mathbb{B}}}"] & \\
\overline{X}^{\mathbb{A}} \arrow[rr, "f \circ m_{\overline{X}^{\mathbb{A}}}"'] & & B
\end{tikzcd}$$

both $\eta^{\mathbb{A}}_{m_X}$ and $\phi_{m_X} \circ \eta^{\mathbb{B}}_{f_\star(m_X)}$ are solutions to the unique diagonal below:

$$\begin{tikzcd}
X \arrow[r, two heads, "1"] \arrow[d, "e_{\overline{X}^{\mathbb{A}}} \circ \eta_X"'] & X \arrow[d, "f \circ m_X"] \arrow[dl, dashed] \\
\overline{X}^{\mathbb{A}} \arrow[r, hook, "f \circ m_{\overline{X}^{\mathbb{A}}}"'] & B
\end{tikzcd}.$$

◀

▶ **Lemma 15.** *There exists a monad morphism* $(U, \alpha) : \overline{(\cdot)}^{\mathbb{X}} \to T$.

Proof. We propose the following definition:

$$\alpha : T \circ U \Rightarrow U \circ \overline{(\cdot)}^{\mathbb{X}} \qquad \alpha_{m_Y} := e_{\overline{Y}} : TY \to \overline{Y}.$$

From the definition of $\overline{(\cdot)}^{\mathbb{X}}$ on morphisms it follows that α is a natural transformation. The commutative diagrams turning (U, α) into a morphism between monads correspond to the following two equations:

$$\mu^{\mathbb{X}}_{m_{\overline{Y}}} \circ e_{\overline{\overline{Y}}} \circ Te_{\overline{Y}} = e_{\overline{Y}} \circ \mu_Y \qquad \eta^{\mathbb{X}}_{m_Y} = e_{\overline{Y}} \circ \eta_Y .$$

Above equalities are satisfied by the definitions of $\eta^{\mathbb{X}}$ and $\mu^{\mathbb{X}}$, respectively. ◀

▶ **Lemma 16.** *Figure 5b commutes for any algebra homomorphism* $f : \mathbb{A} \to \mathbb{B} \in \mathcal{M}$.

Proof. We compose monad morphisms as defined in [36]. We have to show that the monad morphism $(U_{\mathbb{B}} \circ f_*, \beta) : \overline{(\cdot)}^{\mathbb{A}} \to T$ with the natural transformation

$$\beta = T \circ (U_{\mathbb{B}} \circ f_*) \overset{\alpha_{\mathbb{B}} \circ f_*}{\Longrightarrow} U_{\mathbb{B}} \circ \overline{(\cdot)}^{\mathbb{B}} \circ f_* \overset{U_{\mathbb{B}} \circ \alpha_f}{\Longrightarrow} (U_{\mathbb{B}} \circ f_*) \circ \overline{(\cdot)}^{\mathbb{A}}$$

given on a subobject $m_Y : Y \to A$ in $\mathrm{Sub}(A)$ by the morphism

$$\beta_{m_Y} : TY \xrightarrow{e_{\overline{Y}^{\mathbb{B}}}} \overline{Y}^{\mathbb{B}} \xrightarrow{\phi_{m_Y}} \overline{Y}^{\mathbb{A}}$$

coincides with the monad morphism $(U_{\mathbb{A}}, \alpha_{\mathbb{A}}) : \overline{(\cdot)}^{\mathbb{A}} \to T$ with $(\alpha_{\mathbb{A}})_{m_Y} = e_{\overline{Y}^{\mathbb{A}}}$.

The equality $U_{\mathbb{B}} \circ f_\star = U_{\mathbb{A}}$ is immediate from the involved definitions. The identity $\phi_{m_Y} \circ e_{\overline{Y}^{\mathbb{B}}} = e_{\overline{Y}^{\mathbb{A}}}$ follows from the definition of ϕ_{m_Y} as unique diagonal. ◀

▶ **Lemma 17.** $\overline{m_{\mathrm{im}(f)}}^{\mathbb{X}} = m_{\mathrm{im}(f^\sharp)}$ *in* $\mathrm{Sub}(\mathbb{X})$.

Proof. Using the factorisation of $(m_{\mathrm{im}(f)})^\sharp = h_X \circ Tm_{\mathrm{im}(f)}$ in $\mathrm{Alg}(T)$,

$$(m_{\mathrm{im}(f)})^\sharp = (T\mathrm{im}(f), \mu_{\mathrm{im}(f)}) \overset{e_{\overline{\mathrm{im}(f)}}}{\twoheadrightarrow} (\overline{\mathrm{im}(f)}, h_{\overline{\mathrm{im}(f)}}) \overset{m_{\overline{\mathrm{im}(f)}}}{\hookrightarrow} (X, h_X)$$

one easily verifies that the diagram below commutes:

$$\begin{tikzcd}
(TY, \mu_Y) \arrow[rrr, two heads, "e_{\mathrm{im}(f^\sharp)}"] \arrow[d, two heads, "Te_{\mathrm{im}(f)}"'] \arrow[dr, "Tf"] & & & (\mathrm{im}(f^\sharp), h_{\mathrm{im}(f^\sharp)}) \arrow[dd, hook, "m_{\mathrm{im}(f^\sharp)}"] \\
(T\mathrm{im}(f), \mu_{\mathrm{im}(f)}) \arrow[r, "Tm_{\mathrm{im}(f)}"'] \arrow[d, two heads, "e_{\overline{\mathrm{im}(f)}}"'] & (TX, \mu_X) \arrow[drr, "h_X"] & & \\
(\overline{\mathrm{im}(f)}, h_{\overline{\mathrm{im}(f)}}) \arrow[rrr, hook, "m_{\overline{\mathrm{im}(f)}}"'] & & & (X, h_X)
\end{tikzcd} .$$

Since factorisations are unique up to unique isomorphism, there thus exists a unique isomorphism $\phi : m_{\mathrm{im}(f^\sharp)} \simeq m_{\overline{\mathrm{im}(f)}}$ of subobjects of $\mathbb{X}$ as indicated below:

$$\begin{tikzcd}
(TY, \mu_Y) \arrow[r, two heads, "e_{\mathrm{im}(f^\sharp)}"] \arrow[d, two heads, "e_{\overline{\mathrm{im}(f)}} \circ Te_{\mathrm{im}(f)}"'] & (\mathrm{im}(f^\sharp), h_{\mathrm{im}(f^\sharp)}) \arrow[d, hook, "m_{\mathrm{im}(f^\sharp)}"] \arrow[dl, dashed, "\phi"] \\
(\overline{\mathrm{im}(f)}, h_{\overline{\mathrm{im}(f)}}) \arrow[r, hook, "m_{\overline{\mathrm{im}(f)}}"'] & (X, h_X)
\end{tikzcd} .$$

Since by definition $\overline{m_{\mathrm{im}(f)}}^{\mathbb{X}} \simeq m_{\overline{\mathrm{im}(f)}}$, this shows the claim. ◀

▶ **Lemma 20.** *There exists a free-forgetful adjunction* $\mathrm{GAlg}(T) \dashv \mathrm{Alg}(T)$*.*

Proof. We define the forgetful functor $U : \mathrm{GAlg}(T) \to \mathrm{Alg}(T)$ as projection on the first component, i.e. $U(\mathbb{X}_\alpha, \alpha) := \mathbb{X}_\alpha$ and $U(f, p) := f$. For the functor $F : \mathrm{Alg}(T) \to \mathrm{GAlg}(T)$ we propose the definition

$$F(\mathbb{X}) := (\mathbb{X}, (X, \mathrm{id}_X, \eta_X)) \quad \text{and} \quad F(f : \mathbb{X} \to \mathbb{Y}) := (f, \eta_Y \circ f).$$

Since every algebra can be generated by itself, the definition for F is well-defined on objects. For morphisms, one easily establishes (2) from the naturality of η, the monad law $\mu_Y \circ T\eta_Y = \mathrm{id}_{TY}$, and the commutativity of f with algebra structures. The compositionality of F follows analogously; preservation of identity is trivial. For the natural isomorphism

$$\mathrm{Hom}_{\mathrm{GAlg}(T)}(F(\mathbb{X}), (\mathbb{X}_\alpha, \alpha)) \simeq \mathrm{Hom}_{\mathrm{Alg}(T)}(\mathbb{X}, U(\mathbb{X}_\alpha, \alpha))$$

we propose mapping (f, p) to f, and conversely, f to $(f, d_\alpha \circ f)$. The latter is well-defined since

$$(d_\alpha \circ f)^\sharp \circ \eta_X = d_\alpha \circ f \quad \text{and} \quad i_\alpha^\sharp \circ (d_\alpha \circ f)^\sharp = i_\alpha^\sharp \circ d_\alpha \circ f^\sharp = f^\sharp = f \circ (\mathrm{id}_X)^\sharp.$$

Composition in one of the directions trivially yields the identity; for the other direction we note that if (f, p) satisfies (2), then $p = p^\sharp \circ \eta_X = d_\alpha \circ f$. ◀

▶ **Lemma 21.** *Let T be a monoidal monad on $(\mathcal{C}, \otimes, I)$ satisfying (A1) and (A2). Let $\alpha = (Y_\alpha, i_\alpha, d_\alpha)$ and $\beta = (Y_\beta, i_\beta, d_\beta)$ be generators (bases) for T-algebras $\mathbb{X}_\alpha$ and $\mathbb{X}_\beta$. Then $\alpha \boxtimes \beta = (Y_\alpha \otimes Y_\beta, \iota_{\mathbb{X}_\alpha, \mathbb{X}_\beta} \circ (i_\alpha \otimes i_\beta), (d_\alpha \boxtimes d_\beta))$ is a generator (basis) for the T-algebra $\mathbb{X}_\alpha \boxtimes \mathbb{X}_\beta$.*

Proof. First, we calculate

$$\begin{aligned}
& h_\alpha \boxtimes h_\beta \\
= {} & (\mathrm{id} = \mu_{X_\alpha \otimes X_\beta} \circ T(\eta_{X_\alpha \otimes X_\beta})) \\
& (h_\alpha \boxtimes h_\beta) \circ \mu_{X_\alpha \otimes X_\beta} \circ T(\eta_{X_\alpha \otimes X_\beta}) \\
= {} & (q_{\mathbb{X}_\alpha, \mathbb{X}_\beta} = h_\alpha \boxtimes h_\beta \text{ [34]}) \\
& q_{\mathbb{X}_\alpha, \mathbb{X}_\beta} \circ \mu_{X_\alpha \otimes X_\beta} \circ T(\eta_{X_\alpha \otimes X_\beta}) \\
= {} & (q_{\mathbb{X}_\alpha, \mathbb{X}_\beta} \text{ is algebra homomorphism}) \\
& h_{\alpha \boxtimes \beta} \circ T(q_{\mathbb{X}_\alpha, \mathbb{X}_\beta}) \circ T(\eta_{X_\alpha \otimes X_\beta}) \\
= {} & (\text{Definition of } \iota_{\mathbb{X}_\alpha, \mathbb{X}_\beta}) \\
& h_{\alpha \boxtimes \beta} \circ T(\iota_{\mathbb{X}_\alpha, \mathbb{X}_\beta}).
\end{aligned} \tag{9}$$

If α and β are generators, it thus follows

$$\begin{aligned}
&h_{\alpha\boxtimes\beta} \circ T(\iota_{\mathbb{X}_\alpha,\mathbb{X}_\beta}) \circ T(i_\alpha \otimes i_\beta) \circ (d_\alpha \boxtimes d_\beta) \\
= {} & (9) \\
&(h_\alpha \boxtimes h_\beta) \circ T(i_\alpha \otimes i_\beta) \circ (d_\alpha \boxtimes d_\beta) \\
= {} & (T(f \otimes g) = Tf \boxtimes Tg \text{ [34]}) \\
&(h_\alpha \boxtimes h_\beta) \circ (T(i_\alpha) \boxtimes T(i_\beta)) \circ (d_\alpha \boxtimes d_\beta) \\
= {} & (\boxtimes \text{ is functorial}) \\
&(h_\alpha \circ T(i_\alpha) \circ d_\alpha) \boxtimes (h_\beta \circ T(i_\beta) \circ d_\beta) \\
= {} & (\alpha, \beta \text{ are generators}) \\
&\mathrm{id}_{X_\alpha} \boxtimes \mathrm{id}_{X_\beta} \\
= {} & (\boxtimes \text{ is functorial}) \\
&\mathrm{id}_{X_\alpha \boxtimes X_\beta}.
\end{aligned}$$

The additional equality for the case in which α and β are bases follows analogously. ◀

▶ **Corollary 22.** *Let T be a monoidal monad on $(\mathcal{C}, \otimes, I)$ such that (A1) and (A2) are satisfied. The definitions $(\mathbb{X}_\alpha, \alpha) \boxtimes (\mathbb{X}_\beta, \beta) := (\mathbb{X}_\alpha \boxtimes \mathbb{X}_\beta, \alpha \boxtimes \beta)$ and $(f, p) \boxtimes (g, q) := (f \boxtimes g, T_{Y_{\alpha'}, Y_{\beta'}} \circ (p \otimes q))$ yield monoidal structures with unit $((TI, \mu_I), (I, \eta_I, \mathrm{id}_{TI}))$ on $\mathrm{GAlg}(T)$ and $\mathrm{BAlg}(T)$.*

Proof. By Lemma 21 the construction is well-defined on objects. Its well-definedness on morphisms, i.e. the commutativity of (2), is a consequence of the equalities $Tf \boxtimes Tg = T(f \otimes g)$ and $q_{\mathbb{X}_\alpha, \mathbb{X}_\beta} = h_\alpha \boxtimes h_\beta$ [34], which imply $(T_{Y_\alpha, Y_\beta} \circ (p \otimes q))^\sharp = (\mu_{Y_{\alpha'}} \boxtimes \mu_{Y_{\beta'}}) \circ (Tp \boxtimes Tq)$. The natural isomorphisms underlying the monoidal structure for $\mathrm{Alg}(T)$ can be extended to $\mathrm{GAlg}(T)$ by associating canonical Kleisli-morphisms between generators as in (3). ◀

▶ **Lemma 42** ([45])**.** *Let (Y, i, d) be a basis for a T-algebra (X, h). Then $\mu_Y \circ Td = d \circ h$ and $d \circ i = \eta_Y$.*

▶ **Lemma 25.** *The function (4) is a T-algebra homomorphism $p^{\alpha\beta} : \mathbb{X}_\alpha \to \mathbb{X}_\beta$.*

Proof. Using Lemma 42 we deduce the commutativity of the following diagram:

$$\begin{tikzcd}
TX_\alpha \arrow[r, "Td_\alpha"] \arrow[d, "h_\alpha"'] & T^2Y_\alpha \arrow[r, "T^2p"] \arrow[d, "\mu_{Y_\alpha}"] & T^3Y_\beta \arrow[r, "T\mu_{Y_\beta}"] \arrow[d, "\mu_{TY_\beta}"] & T^2Y_\beta \arrow[r, "T^2i_\beta"] \arrow[d, "\mu_{Y_\beta}"] & T^2X_\beta \arrow[r, "Th_\beta"] \arrow[d, "\mu_{X_\beta}"] & TX_\beta \arrow[d, "h_\beta"] \\
X_\alpha \arrow[r, "d_\alpha"] & TY_\alpha \arrow[r, "Tp"] & T^2Y_\beta \arrow[r, "\mu_{Y_\beta}"] & TY_\beta \arrow[r, "Ti_\beta"] & TX_\beta \arrow[r, "h_\beta"] & X_\beta
\end{tikzcd}.$$

◀

▶ **Lemma 26.** *The operations (3) and (4) are mutually inverse.*

Proof. Essentially, the statement follows from the observation that, for bases, the functions involved in the composition below are isomorphisms:

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{Alg}(T)}(\mathbb{X}_\alpha, \mathbb{X}_\beta) \overset{(d_\beta)_* \circ (i_\alpha^\sharp)^*}{\longrightarrow} \mathrm{Hom}_{\mathrm{Alg}(T)}((TY_\alpha, \mu_{Y_\alpha}), (TY_\beta, \mu_{Y_\beta})) \\
&\overset{(\eta_{Y_\alpha})^*}{\longrightarrow} \mathrm{Hom}_{\mathrm{Kl}(T)}(Y_\alpha, Y_\beta). \qquad (10)
\end{aligned}$$

More concretely, the definitions imply

$$(p^{\alpha\beta})_{\alpha\beta} = d_\beta \circ (h_\beta \circ Ti_\beta \circ \mu_{Y_\beta} \circ Tp \circ d_\alpha) \circ i_\alpha$$
$$(f_{\alpha\beta})^{\alpha\beta} = h_\beta \circ Ti_\beta \circ \mu_{Y_\beta} \circ T(d_\beta \circ f \circ i_\alpha) \circ d_\alpha.$$

Using Lemma 42 we deduce the commutativity of the diagrams below

$$\begin{tikzcd}
Y_\alpha \arrow[rrr, "p"] \arrow[d, "i_\alpha"'] \arrow[dr, "\eta_{Y_\alpha}"] & & & TY_\beta \arrow[rr, "\mathrm{id}_{TY_\beta}"] \arrow[dl, "\eta_{TY_\beta}"] \arrow[d, "\mathrm{id}_{TY_\beta}"] & & TY_\beta \\
X_\alpha \arrow[r, "d_\alpha"] & TY_\alpha \arrow[r, "Tp"] & T^2Y_\beta \arrow[r, "\mu_{Y_\beta}"] & TY_\beta \arrow[r, "Ti_\beta"] & TX_\beta \arrow[r, "h_\beta"] & X_\beta \arrow[u, "d_\beta"']
\end{tikzcd}$$

$$\begin{tikzcd}
X_\alpha \arrow[r, "\mathrm{id}_{X_\alpha}"] \arrow[d, "Ti_\alpha \circ d_\alpha"'] & X_\alpha \arrow[rr, "f"] & & X_\beta \arrow[r, "\mathrm{id}_{X_\beta}"] \arrow[d, "d_\beta"] & X_\beta \\
TX_\alpha \arrow[r, "Tf"] \arrow[ur, "h_\alpha"] & TX_\beta \arrow[r, "Td_\beta"] \arrow[urr, "h_\beta"] & T^2Y_\beta \arrow[r, "\mu_{Y_\beta}"] & TY_\beta \arrow[r, "Ti_\beta"] & TX_\beta \arrow[u, "h_\beta"']
\end{tikzcd} .$$

◀

▶ **Lemma 27.** *$f_{\alpha\beta}$ is the unique Kleisli-morphism such that $f_{\alpha\beta} \cdot d_\alpha = d_\beta \circ f$. Conversely, $p^{\alpha\beta}$ is the unique T-algebra homomorphism such that $p \cdot d_\alpha = d_\beta \circ p^{\alpha\beta}$.*

Proof. The definitions imply

$$f_{\alpha\beta} \cdot d_\alpha = \mu_{Y_\beta} \circ T(d_\beta \circ f \circ i_\alpha) \circ d_\alpha.$$

Using Lemma 42 we deduce the commutativity of the diagram below:

$$\begin{tikzcd}
X_\alpha \arrow[r, "d_\alpha"] \arrow[d, "\mathrm{id}_{X_\alpha}"'] & TY_\alpha \arrow[r, "Ti_\alpha"] & TX_\alpha \arrow[r, "Tf"] \arrow[dll, "h_\alpha"] & TX_\beta \arrow[r, "Td_\beta"] \arrow[dll, "h_\beta"] & T^2Y_\beta \arrow[d, "\mu_{Y_\beta}"] \\
X_\alpha \arrow[r, "f"'] & X_\beta \arrow[rrr, "d_\beta"'] & & & TY_\beta
\end{tikzcd} .$$

Since an equality of the type $p \cdot d_\alpha = d_\beta \circ f$ implies

$$p = \mu_{Y_\beta} \circ \eta_{TY_\beta} \circ p = \mu_{Y_\beta} \circ Tp \circ \eta_{Y_\alpha} = \mu_{Y_\beta} \circ Tp \circ d_\alpha \circ i_\alpha = d_\beta \circ f \circ i_\alpha = f_{\alpha\beta},$$

the morphism $f_{\alpha\beta}$ is moreover uniquely determined. For the second part of the claim we observe that by above and Lemma 26 it holds $p \cdot d_\alpha = (p^{\alpha\beta})_{\alpha\beta} \cdot d_\alpha = d_\beta \circ p^{\alpha\beta}$, and that an equality of the type $p \cdot d_\alpha = d_\beta \circ f$ implies $p^{\alpha\beta} = i_\beta^\sharp \circ (p \cdot d_\alpha) = i_\beta^\sharp \circ d_\beta \circ f = f$. ◀

▶ **Lemma 43.** *$g_{\beta\gamma} \cdot f_{\alpha\beta} = (g \circ f)_{\alpha\gamma}$.*

Proof. The definitions imply

$$g_{\beta\gamma} \cdot f_{\alpha\beta} = \mu_{Y_\gamma} \circ T(d_\gamma \circ g \circ i_\beta) \circ d_\beta \circ f \circ i_\alpha$$
$$(g \circ f)_{\alpha\gamma} = d_\gamma \circ (g \circ f) \circ i_\alpha.$$

We delete common terms and use Lemma 42 to deduce the commutativity of the diagram below:

$$\begin{tikzcd}
X_\beta \arrow[r, "d_\beta"] \arrow[d, "\mathrm{id}_{X_\beta}"'] & TY_\beta \arrow[r, "Ti_\beta"] & TX_\beta \arrow[r, "Tg"] \arrow[dll, "h_\beta"] & TX_\gamma \arrow[r, "Td_\gamma"] \arrow[dll, "h_\gamma"] & T^2Y_\gamma \arrow[d, "\mu_{Y_\gamma}"] \\
X_\beta \arrow[r, "g"'] & X_\gamma \arrow[rrr, "d_\gamma"'] & & & TY_\gamma
\end{tikzcd} .$$

◀

▶ **Lemma 44.** *$q^{\beta\gamma} \circ p^{\alpha\beta} = (q \cdot p)^{\alpha\gamma}$.*

Proof. The definitions imply

$$\begin{aligned} q^{\beta\gamma} \circ p^{\alpha\beta} &= (h_\gamma \circ Ti_\gamma \circ \mu_{Y_\gamma} \circ Tq \circ d_\beta) \circ (h_\beta \circ Ti_\beta \circ \mu_{Y_\beta} \circ Tp \circ d_\alpha) \\ (q \cdot p)^{\alpha\gamma} &= h_\gamma \circ Ti_\gamma \circ \mu_{Y_\gamma} \circ T\mu_{Y_\gamma} \circ T^2 q \circ Tp \circ d_\alpha. \end{aligned}$$

By deleting common terms and using the equality $d_\beta \circ h_\beta \circ Ti_\beta = \mathrm{id}_{TY_\beta}$ it is thus sufficient to show

$$\mu_{Y_\gamma} \circ Tq \circ \mu_{Y_\beta} = \mu_{Y_\gamma} \circ T\mu_{Y_\gamma} \circ T^2 q.$$

Above equation follows from the commutativity of the diagram below:

$$\begin{tikzcd} T^2Y_\beta \arrow[r, "\mu_{Y_\beta}"] \arrow[d, "T^2q"'] & TY_\beta \arrow[r, "Tq"] & T^2Y_\gamma \arrow[d, "\mu_{Y_\gamma}"] \\ T^3Y_\gamma \arrow[rru, "\mu_{TY_\gamma}"] \arrow[r, "T\mu_{Y_\gamma}"'] & T^2Y_\gamma \arrow[r, "\mu_{Y_\gamma}"'] & TY_\gamma \end{tikzcd} .$$

◀

▶ **Lemma 28.** *$g_{\beta\gamma} \cdot f_{\alpha\beta} = (g \circ f)_{\alpha\gamma}$ and $q^{\beta\gamma} \circ p^{\alpha\beta} = (q \cdot p)^{\alpha\gamma}$.*

Proof. Follows from Lemma 43 and Lemma 44. ◀

▶ **Corollary 29.** *There exist isomorphisms of categories $\mathrm{BAlg}(T) \simeq \mathrm{Alg}_{\mathrm{B}}(T) \simeq \mathrm{Kl}_{\mathrm{B}}(T)$.*

Proof. For the first isomorphism we define a functor $F : \mathrm{BAlg}(T) \to \mathrm{Alg}_{\mathrm{B}}(T)$ by $F(\mathbb{X}_\alpha, \alpha) = (\mathbb{X}_\alpha, \alpha)$ and $F(f, p) = f$; and a functor $G : \mathrm{Alg}_{\mathrm{B}}(T) \to \mathrm{BAlg}(T)$ by $G(\mathbb{X}_\alpha, \alpha) = (\mathbb{X}_\alpha, \alpha)$ and $G(f) := (f, f_{\alpha\beta})$. Well-definedness and mutual invertibility are consequences of Lemma 43, and Lemma 27, respectively. For the second isomorphism we define a functor $F : \mathrm{Alg}_{\mathrm{B}}(T) \to \mathrm{Kl}_{\mathrm{B}}(T)$ by $F(\mathbb{X}_\alpha, \alpha) = (\mathbb{X}_\alpha, \alpha)$ and $Ff = f_{\alpha\beta}$; and a functor $G : \mathrm{Kl}_{\mathrm{B}}(T) \to \mathrm{Alg}_{\mathrm{B}}(T)$ by $G(\mathbb{X}_\alpha, \alpha) = (\mathbb{X}_\alpha, \alpha)$ and $Gp = p^{\alpha\beta}$. Well-definedness and mutual invertibility are consequences of Lemma 43, Lemma 44, and Lemma 26, respectively. ◀

▶ **Proposition 30.** *There exist Kleisli isomorphisms p and q such that $f_{\alpha'\beta'} = q \cdot f_{\alpha\beta} \cdot p$.*

Proof. The Kleisli morphisms p and q and their respective candidates for inverses p^{-1} and q^{-1} are defined below

$$\begin{aligned} p \; &:= d_\alpha \circ i_{\alpha'} : Y_{\alpha'} \longrightarrow TY_\alpha & \qquad q \; &:= d_{\beta'} \circ i_\beta : Y_\beta \longrightarrow TY_{\beta'} \\ p^{-1} &:= d_{\alpha'} \circ i_\alpha : Y_\alpha \longrightarrow TY_{\alpha'} & \qquad q^{-1} &:= d_\beta \circ i_{\beta'} : Y_{\beta'} \longrightarrow TY_\beta. \end{aligned}$$

From Lemma 42 it follows that the diagram below commutes:xr

$$\begin{tikzcd} Y_\alpha \arrow[r, "i_\alpha"] \arrow[dd, "\eta_{Y_\alpha}"'] & X_\alpha \arrow[r, "d_{\alpha'}"] \arrow[d, "\mathrm{id}_{X_\alpha}"] & TY_{\alpha'} \arrow[dd, "Ti_{\alpha'}"] \\ & X_\alpha \arrow[dl, "d_\alpha"] & \\ TY_\alpha & T^2Y_\alpha \arrow[l, "\mu_{Y_\alpha}"] & TX_\alpha \arrow[l, "Td_\alpha"] \arrow[ul, "h_\alpha"'] \end{tikzcd} .$$

This shows that p^{-1} is a Kleisli right-inverse of p. A symmetric version of above diagram shows that p^{-1} is also a Kleisli left-inverse of p. Analogously it follows that q^{-1} is a Kleisli inverse of q.

The definitions further imply the equalities

$$\begin{aligned} q \cdot f_{\alpha\beta} \cdot p &= \mu_{Y_{\beta'}} \circ T(d_{\beta'} \circ i_\beta) \circ \mu_{Y_\beta} \circ T(d_\beta \circ f \circ i_\alpha) \circ d_\alpha \circ i_{\alpha'} \\ f_{\alpha'\beta'} &= d_{\beta'} \circ f \circ i_{\alpha'}. \end{aligned}$$

We delete common terms and use Lemma 42 to establish the commutativity of the diagram below:

$$\begin{tikzcd}
X_\alpha \arrow[r, "d_\alpha"] \arrow[dr, "\mathrm{id}_{X_\alpha}"] \arrow[dd, "f"'] & TY_\alpha \arrow[r, "Ti_\alpha"] & TX_\alpha \arrow[r, "Tf"] \arrow[dl, "h_\alpha"'] & TX_\beta \arrow[d, "Td_\beta"] \arrow[ddll, "h_\beta"] \\
 & X_\alpha \arrow[d, "f"] & & T^2Y_\beta \arrow[dd, "\mu_{Y_\beta}"] \\
X_\beta \arrow[d, "d_{\beta'}"'] & X_\beta \arrow[l, "\mathrm{id}_{X_\beta}"'] \arrow[drr, "d_\beta"] & & \\
TY_{\beta'} & T^2Y_{\beta'} \arrow[l, "\mu_{Y_{\beta'}}"] & TX_\beta \arrow[l, "Td_{\beta'}"] \arrow[ull, "h_\beta"'] & TY_\beta \arrow[l, "Ti_\beta"]
\end{tikzcd}.$$

◀

▶ **Proposition 31.** *There exists a Kleisli isomorphism p with Kleisli inverse p^{-1} such that $f_{\alpha'\alpha'} = p^{-1} \cdot f_{\alpha\alpha} \cdot p$.*

Proof. In Proposition 30 let $\beta = \alpha$ and $\beta' = \alpha'$. One verifies that in the corresponding proof the definitions of the morphisms p^{-1} and q coincide. ◀

▶ **Lemma 32.** *Let (Y, k_Y, i, d) be a generator for (X, h, k). Then $i^\sharp : TY \to X$ is a λ-bialgebra homomorphism $i^\sharp : \mathrm{free}_T(Y, k_Y) \to (X, h, k)$.*

Proof. By definition we have $\mathrm{free}(Y, k_Y) = (TY, \mu_Y, \lambda_Y \circ Tk_Y)$. Clearly the lifting $i^\sharp = h \circ Ti$ is a T-algebra homomorphism $i^\sharp : (TY, \mu_Y) \to (X, h)$. It is a F-coalgebra homomorphism $i^\sharp : (TY, \lambda_Y \circ Tk_Y) \to (X, k)$ since the diagram below commutes:

$$\begin{tikzcd}
TY \arrow[r, "Ti"] \arrow[d, "Tk_Y"'] & TX \arrow[r, "h"] \arrow[d, "Tk"] & X \arrow[dd, "k"] \\
TFY \arrow[r, "TFi"] \arrow[d, "\lambda_Y"'] & TFX \arrow[d, "\lambda_X"] & \\
FTY \arrow[r, "FTi"] & FTX \arrow[r, "Fh"] & FX
\end{tikzcd}.$$

◀

▶ **Lemma 33.** *Let (Y, k_Y, i, d) be a basis for (X, h, k), then $\mathrm{free}_T(Y, k_Y) = \mathrm{exp}_T(Y, Fd \circ k \circ i)$.*

Proof. Using Lemma 42 we establish the commutativity of the diagram below:

$$\begin{tikzcd}
TY \arrow[rr, "Tk_Y"] \arrow[dd, "\mathrm{id}_{TY}"'] & & TFY \arrow[rr, "\lambda_Y"] \arrow[dd, "TFi"] & & FTY \arrow[rr, "\mathrm{id}_{FTY}"] \arrow[d, "FTi"] & & FTY \arrow[dd, "\mathrm{id}_{FTY}"] \\
 & & & & FTX \arrow[r, "Fh"] \arrow[d, "FTd"] & FX \arrow[dr, "Fd"] & \\
TY \arrow[r, "Ti"] & TX \arrow[r, "Tk"] & TFX \arrow[r, "TFd"] & TFTY \arrow[r, "\lambda_{TY}"] & FT^2Y \arrow[rr, "F\mu_Y"] & & FTY
\end{tikzcd}.$$

◀

▶ **Lemma 35.** *Let (X, h, k) be a λ-bialgebra and (Y, i, d) a basis for the T-algebra (X, h). Then $(TY, (Fd \circ k \circ i)^\sharp, i^\sharp, \eta_{TY} \circ d)$ is a generator for (X, h, k).*

Proof. In the following we abbreviate $k_{TY} := (Fd \circ k \circ i)^\sharp = F\mu_Y \circ \lambda_{TY} \circ T(Fd \circ k \circ i)$. By Proposition 6 the lifting $i^\sharp$ is a F-coalgebra homomorphism $i^\sharp : (TY, k_{TY}) \to (X, k)$. This shows the commutativity of the diagram on the left of (5). By [45, Prop. 15] the morphism d is a F-coalgebra homomorphism in the reverse direction. Together with the commutativity of the diagram on the left below this implies the commutativity of the second diagram to the left of (5):

$$\begin{tikzcd}
TY \arrow[r, "\eta_{TY}"] \arrow[dd, "k_{TY}"'] & T^2Y \arrow[d, "Tk_{TY}"] \\
 & TFTY \arrow[d, "\lambda_{TY}"] \\
FTY \arrow[ur, "\eta_{FTY}"] \arrow[r, "F\eta_{TY}"] & FT^2Y
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2Y \arrow[r, "T^2i"] & T^2X \arrow[rr, "Th"] \arrow[dr, "\mu_X"] & & TX \arrow[dd, "h"] \\
TY \arrow[u, "\eta_{TY}"] \arrow[r, "Ti"] & TX \arrow[u, "\eta_{TX}"'] \arrow[r, "\mathrm{id}_{TX}"] & TX \arrow[dr, "h"] & \\
X \arrow[u, "d"] \arrow[rrr, "\mathrm{id}_X"] & & & X
\end{tikzcd}
.$$

Similarly, the commutativity of third diagram to the left of (5) follows from the commutativity of the diagram on the right above. ◀

▶ **Lemma 36.** *Let (Y, i, d) be a basis for a T-algebra (X, h), then (6) commutes for $k := Ti \circ d$.*

Proof. The commutativity of the diagram on the left of (6) follows from Lemma 42 and the naturality of μ. The diagram in the middle of (6) commutes by the definition of a generator. The commutativity of the diagram on the right of (6) is again a consequence of Lemma 42:

$$\begin{tikzcd}
X \arrow[r, "d"] \arrow[d, "d"'] & TY \arrow[r, "Ti"] & TX \arrow[dd, "T\eta_X"] \\
TY \arrow[ur, "\mathrm{id}_{TY}"] \arrow[d, "Ti"'] \arrow[dr, "T\eta_Y"] & & \\
TX \arrow[r, "Td"] & T^2Y \arrow[r, "T^2i"] & T^2X
\end{tikzcd}
.$$

◀

▶ **Lemma 37.** *Let (Y, i, d) be a basis for a T-algebra (X, h) and $k := Ti \circ d$. Then $\eta_X \circ i = k \circ i$ and $Tk \circ (\eta_X \circ i) = T\eta_X \circ (\eta_X \circ i)$.*

Proof. The statement follows from Lemma 42:

$$\begin{tikzcd}
Y \arrow[rr, "i"] \arrow[d, "i"] \arrow[dr, "\eta_Y"] & & X \arrow[d, "\eta_X"] \\
X \arrow[r, "d"] & TY \arrow[r, "Ti"] & TX
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "i"] \arrow[dd, "i"] \arrow[dr, "\eta_Y"] & X \arrow[r, "\eta_X"] \arrow[d, "d"] & TX \arrow[r, "Td"] & T^2Y \arrow[dd, "T^2i"] \\
 & TY \arrow[urr, "\eta_{TY}"] \arrow[d, "Ti"] & & \\
X \arrow[r, "\eta_X"] \arrow[rrr, bend right, "T\eta_X \circ \eta_X"'] & TX \arrow[rr, "\eta_{TX}"] & & T^2X
\end{tikzcd}
.$$

◀

▶ **Corollary 38.** *Let* $\alpha := (Y, i, d)$ *be a non-empty basis for a set-based* T*-algebra* (X, h) *and* $k := Ti \circ d$*. Then* $(\mathrm{id}_{(X,h)})_{\alpha, GF\alpha} : Y \to TY_k$ *is the unique morphism making the diagram below commute:*

$$\begin{tikzcd} Y \arrow[r, dashed] \arrow[rr, bend left, "\eta_X \circ i"] & TY_k \arrow[r, "Ti_k"'] & TX \arrow[r, shift left, "Tk"] \arrow[r, shift right, "T\eta_X"'] & T^2X \end{tikzcd} .$$

Proof. Since i_k is an equaliser of k and η_X [18], it follows from Lemma 37 that there exists a unique morphism $\varphi : Y \to Y_k$ such that $i_k \circ \varphi = i$. Since Ti_k is an equaliser of Tk and $T\eta_X$ [18], it follows from Lemma 37 that there exists a unique morphism $\psi : Y \to TY_k$ such that $Ti_k \circ \psi = \eta_X \circ i$. It is not hard to see that $\psi = \eta_{Y_k} \circ \varphi$. The statement thus follows from $(\mathrm{id}_{(X,h)})_{\alpha, GF\alpha} = d_k \circ i = d_k \circ i_k \circ \varphi = \eta_{Y_k} \circ \varphi = \psi$. ◀

▶ **Lemma 39.** *A morphism* $i : Y \to X$ *is part of a generator for a* $T_{\Sigma,E}$*-algebra* $\mathbb{X}$ *if(f) every element of* X *can be expressed as a* Σ*-term in* $i[Y]$ *modulo* E*.*

Proof. First note that the equivalence K preserves the underlying carrier set of an algebra. Under its inverse, the $T_{\Sigma,E}$-algebra homomorphism $i^\sharp$ corresponds to the function $K^{-1}(i^\sharp)$ between $\mathbb{S}_\Sigma$-algebras satisfying E inductively defined by $K^{-1}(i^\sharp)(y) = i(y)$ and $K^{-1}(i^\sharp)(\sigma(t_1, ..., t_{\mathrm{ar}(\sigma)})) = \sigma^{\mathbb{X}}(K^{-1}(i^\sharp)(t_1), ..., K^{-1}(i^\sharp)(t_{\mathrm{ar}(\sigma)}))$. In consequence, the identity $i^\sharp \circ d = \mathrm{id}_{\mathbb{X}}$ thus translates to the observation that any element $x \in K^{-1}(\mathbb{X})$ can be expressed as the Σ-term $K^{-1}(i^\sharp)(t)$ in $i[Y]$, where $t = K^{-1}(d(x))$. ◀